# Aggregation of Nonparametric Estimators for Volatility Matrix *

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**ABSTRACT**

An aggregated method of nonparametric estimators based on time-domain and state-domain estimators is proposed and studied. To attenuate the *curse of dimensionality*, we propose a factor modeling strategy. We first investigate the asymptotic behavior of nonparametric estimators of the volatility matrix in the time domain and in the state domain. Asymptotic normality is separately established for nonparametric estimators in the time domain and state domain. These two estimators are asymptotically independent. Hence, they can be combined, through a dynamic weighting scheme, to improve the efficiency of volatility matrix estimation. The optimal dynamic weights are derived, and it is shown that the aggregated estimator uniformly dominates volatility matrix estimators using time-domain or state-domain smoothing alone. A simulation study, based on an essentially affine model for the term structure, is conducted, and it demonstrates convincingly that the newly proposed procedure outperforms both time- and state-domain estimators. Empirical studies further endorse the advantages of our aggregated method.

**KEYWORDS**: aggregation, nonparametric function estimation, diffusion, volatility matrix, factor, local time, affine model.

Covariance matrices are fundamental for risk management, asset pricing, proprietary trading, and portfolio managements. In forecasting a future event such as the volatility matrix, two pieces of information are frequently consulted. Based on the recent history, one uses a form of local average, such as the moving average, to predict the volatility matrix. This approach localizes in time and uses the smoothness of the volatility matrix as a function of time. It ignores completely the historical information, which is related to the current prediction through a stationarity assumption. On the other hand, one can predict a future event by consulting the historical information with similar scenarios. This approach basically localizes in the state variable and depends on the stationarity assumption. For example, by localizing on a few key financial factors, one can compute the volatility matrix using the historical information. This results in a nonparametric estimate of the volatility matrix using state-domain smoothing. See, for example,

Anderson, Bollerslev and Diebold (2002) for a unified framework of interpreting both parametric and nonparametric approaches for volatility measurement.

The aforementioned two estimators are weakly correlated, as they use data that are quite far apart in time. They can be combined to improve the efficiency of the volatility matrix estimation. This results in an aggregated estimator of the volatility matrix. Three challenges arise in the endeavor: the *curse of dimensionality*, the choice of dynamic weights, and the mathematical complexity.

Due to the curse of dimensionality, surface smoothing techniques are not very useful in practice when there are more than two or three predictor variables. An efficient dimensionality reduction process should be imposed in state-domain estimation. An introduction to some of these approaches, such as *additive modeling, partially linear modeling, modeling with interactions*, and *multiple index models*, can be found in Fan and Yao (2003).

In this paper, we propose a *factor modeling* strategy to reduce the dimensionality in the state domain smoothing. Specifically, to estimate the covariance matrix among several assets, we first find a few factors that capture the main price dynamics of the underlying assets. Regarding the covariance matrix as a smooth function of these factors, the covariance matrix can be computed via localizing on the factors.

Figure 1 here.

Our approach is particularly appealing for the yields of bonds, as they are often highly correlated, which makes the choice of the factors relatively easy. To elucidate our idea, consider the weekly data on the yields of treasury bills and bonds with maturities 1 year, 5 years, and 10 years presented in Figure 1. We choose the 5-year yield process as the single factor. Suppose that the current time is January 14, 2000 and the current interest rate of the 5-year treasury bond is 6.67%, corresponding to time index $t = 1986$. One may estimate the volatility matrix based on the weighted squared differences in the past 104 weeks. This corresponds to time-domain smoothing, using the small vertical stretch of data shown in Figure 1(a). On the other hand, one may also estimate the volatility matrix using the historical data with interest rates approximately 6.67%, say, $6.67\% \pm .20\%$. This corresponds to localizing in state domain and is indicated by the

horizontal bar in Figure 1(a). Figures 1(b) and 1(c) present scatter plots of the yield differences $X_t^{1\text{yr}} - X_{t-1}^{1\text{yr}}$ for the 1-year bill against the yield differences $X_t^{10\text{yr}} - X_{t-1}^{10\text{yr}}$ for the 10-year bond, using respectively the data localizing in the time and state domains. The associated regression lines of the time- and state-domain data are also presented. The scatter plots give two estimates of the conditional correlation and conditional variance of the volatility matrix for the week of $t = 1986$. They are weakly dependent as the two scatter plots use data that are quite far apart in time.

Let $\widehat{\mathbf{\Sigma}}_{T,t}$ and $\widehat{\mathbf{\Sigma}}_{S,t}$ be the estimated volatility matrices based on data localizing in the time and state domains, respectively. For example, they can be the sample covariance matrices for the data presented in Figures 1(b) and 1(c), respectively for $t = 1986$. To fully utilize these two estimators, we introduce a weight $w_t$ and define an aggregated estimator* as

$$\widehat{\mathbf{\Sigma}}_{A,t} = \omega_t \widehat{\mathbf{\Sigma}}_{S,t} + (1 - \omega_t)\widehat{\mathbf{\Sigma}}_{T,t}. \tag{1}$$

The weight function $\omega_t$ is always between 0 and 1, and it can be an adaptive random process which is observable at time $t$. Due to the weak dependence between the original two estimators, the aggregated estimator is always more efficient than either of the time- and state-domain estimators.

An interesting question is the choice of the dynamic weight $\omega_t$. Suppose we have a portfolio with allocation vector $\mathbf{a}$. Then the aggregation method gives us the following estimate of the portfolio variance:

$$\mathbf{a}^T \widehat{\mathbf{\Sigma}}_{A,t} \mathbf{a} = \omega_t \mathbf{a}^T \widehat{\mathbf{\Sigma}}_{S,t} \mathbf{a} + (1 - \omega_t)\mathbf{a}^T \widehat{\mathbf{\Sigma}}_{T,t} \mathbf{a}. \tag{2}$$

Since $\widehat{\mathbf{\Sigma}}_{S,t}$ and $\widehat{\mathbf{\Sigma}}_{T,t}$ are asymptotically independent[2], the optimal weight in terms of

*Ledoit and Wolf (2003) introduce a shrinkage estimator by combining the sample covariance estimator with that derived from the CAPM. Their procedure intends to improve estimated covariance matrix by pulling the sample covariance towards the estimate based on the CAPM. Their basic assumption is that the return vectors are i.i.d. across time. This usually holds approximately when the data are localized in time. In this sense, their estimator can be regarded as a time-domain estimator.

[2]We prove in Section 4 that $\widehat{\mathbf{\Sigma}}_{S,t}$ and $\widehat{\mathbf{\Sigma}}_{T,t}$ are asymptotically independent, and thus they are close to be independent in finite sample. In the following, by "nearly independent" and "almost uncorrelated", we mean the same.

minimizing the variance of $\mathbf{a}^T\widehat{\mathbf{\Sigma}}_{A,t}\mathbf{a}$ is

$$\omega_{\text{opt},t} = \frac{\text{var}(\mathbf{a}^T\widehat{\mathbf{\Sigma}}_{T,t}\mathbf{a})}{\text{var}(\mathbf{a}^T\widehat{\mathbf{\Sigma}}_{S,t}\mathbf{a}) + \text{var}(\mathbf{a}^T\widehat{\mathbf{\Sigma}}_{T,t}\mathbf{a})}. \tag{3}$$

Indeed, our asymptotic result in Section 4 shows that the optimal weight admits a simple and explicit form, independent of $\mathbf{a}$. This makes our implementation very easy.

The above approach is data analytic in the sense that it is always operational. To appreciate our idea, we will introduce a mathematical model for the data-generating process in Section 1. And then in the following several sections we formally show that the aggregated estimator has the desired statistical properties.

# 1 Model and Assumptions

Let $\mathbf{W}_t = (W_1^t, \cdots, W_t^m)^T$ and $\mathbf{W} = \{\mathbf{W}_t,\ \mathcal{F}_t^W;\ 0 \le t < \infty\}$ be an $m$-dimensional standard Brownian motion. Consider the following $d$-dimensional diffusion process

$$d\boldsymbol{X}_t = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{W}_t, \tag{4}$$

where $\boldsymbol{X}_t = (X_t^1, \cdots, X_t^d)^T$, $\boldsymbol{\mu}_t$ is a $d \times 1$ predictable vector process, and $\boldsymbol{\sigma}_t$ is a $d \times m$ predictable matrix process, depending only on $\boldsymbol{X}_t$. Here, $m$ can be different from $d$. This is a widely used model for asset prices and the yields of bonds. This family of models includes famous ones such as multivariate generalizations of both Vasicek (1977) and Cox, Ingersoll and Ross (1985).

Under model (4), the diffusion matrix is $\mathbf{\Sigma}_t = \boldsymbol{\sigma}_t\boldsymbol{\sigma}_t^T$. As mentioned before, when $d \ge 2$, the so-called *curse of dimensionality* makes implementation hard. To reduce the dimensionality, we introduce a scalar factor $f_t$ and model the drift and diffusion processes as $\boldsymbol{\mu}_t = \boldsymbol{\mu}(f_t)$ and $\boldsymbol{\sigma}_t = \boldsymbol{\sigma}(f_t)$, where $\boldsymbol{\mu}(\cdot) = \{\mu_i(\cdot)\}_{1\le i\le d}$ is a $d \times 1$ Borel measurable vector and $\boldsymbol{\sigma}(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i\le d, 1\le j\le m}$ is a $d \times m$ Borel measurable matrix. Then model (4) becomes

$$dX_t^i = \mu_i(f_t)dt + \sum_{j=1}^m \sigma_{ij}(f_t)dW_t^j, \quad 1 \le i \le d. \tag{5}$$

In this model, the diffusion matrix is $\mathbf{\Sigma}(f_t) = \boldsymbol{\sigma}(f_t)\boldsymbol{\sigma}(f_t)^T$. See also Engle, Ng and Rothchild (1990) for a similar strategy.

We introduce some stochastic structure on $f_t$ by assuming that $f_t$ is the solution to the following stochastic differential equation (SDE):

$$df_t = a(f_t)dt + \sum_{j=1}^{m} b_j(f_t)dW_t^j, \tag{6}$$

where $a(\cdot)$ and $b_1(\cdot),\ b_2(\cdot), \cdots,\ b_m(\cdot)$ are unknown functions. In some situations like modeling bond yields[3], the factor $f_t$ can be chosen as one of the bond yields, i.e., $f_t$ is one of the coordinates of $\boldsymbol{X}_t$. But in general, $f_t$ may be different from any coordinate of $\boldsymbol{X}_t$, and the theoretical studies in this paper apply to both cases. The data are observed at times $t_i = t_0 + i\Delta$, $i = 0, 1, \cdots, N$, with sampling interval $\Delta$, resulting in vectors $\{\boldsymbol{X}_{t_i}, i = 0, 1, \cdots, N\}$ and $\{f_{t_i}, i = 0, 1, \cdots, N\}$. This model is reasonable for the yields of bonds with different maturities since they are highly correlated. Thus, localizing on all the yields processes in the state domain results in approximately the same data set as localizing on only one of the yields processes. In addition, our study can be generalized to the multi-factor case without much extra difficulty. We will focus on the one-factor setting for simplicity of presentation.

Let $\boldsymbol{Y}_i = (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i})\Delta^{-1/2}$, and denote by $Y_i^1,\ Y_i^2,\ \cdots,\ Y_i^d$ the coordinates of $\boldsymbol{Y}_i$. Then, by the Euler scheme, we have

$$\boldsymbol{Y}_i \approx \boldsymbol{\mu}(f_{t_i})\sqrt{\Delta} + \boldsymbol{\sigma}(f_{t_i})\boldsymbol{\varepsilon}_{t_i}, \tag{7}$$

where $\boldsymbol{\varepsilon}_{t_i}$ follows the $m$-dimensional standard Gaussian distribution. The conditional covariance matrix of $\boldsymbol{X}$ at time $t_i$ can be approximated by $\Delta\boldsymbol{\Sigma}(f_{t_i})$ (see Fan and Zhang, 2003). Hence, the estimate of the conditional covariance matrix is almost equivalent to the estimate of the diffusion matrix $\boldsymbol{\Sigma}(\cdot)$. Fan and Zhang (2003) study the impact of the order of difference on nonparametric estimation. They found that while higher order can possibly reduce approximation errors, it increases variances of data substantially. They recommended the Euler scheme (7) for most practical situations.

To use time-domain information, it is necessary to assume that the sampling frequency $\Delta$ converges to zero so that the biases in time-domain approximations are negligible. As a result, we face the challenge of developing asymptotic theory for the

[3]In practice, one can take the yields process with median term of maturity as the driving factor, as this bond is highly correlated to both short-term and long-term bonds.

diffusion model (5). Both nonparametric estimators in the time domain and state domain need to be investigated. Pioneering efforts on nonparametric estimation of drift and diffusion include Jacod (1997), Jiang and Knight (1997), Arfi (1998), Gobet (2002), Bandi and Philips (2003), Cai and Hong (2003), Bandi and Moloche (2004), and Chen and Gao(2004). Arapis and Gao (2004) investigate the mean aggregated square errors of several methods for estimating the drift and diffusion, and compare their performances. Aït-Sahalia and Mykland (2003, 2004) study the effects of random and discrete sampling when estimating continuous-time diffusions. Bandi and Nguyen (1999) investigate small sample behaviors of nonparametric diffusion parameters. See Bandi and Phillips (2002) for a survey of recently introduced techniques for identifying nonstationary continuous-time processes. As long as the time horizon is long, the diffusion matrix can be estimated with low frequency data (say, finite $\Delta^{-1}$). See, for example, Hansen *et al.* (1998) for the spectral method, Kessler and Sørensen (1999) for parametric models, and Gobet *et al.* (2004) for specific univariate nonparametric diffusions.

To facilitate our future presentation, we make the following assumptions:

***Assumption 1***. (Global Lipschitz and linear growth conditions). There exists a constant $k_0 \geq 0$ such that

$$\|\boldsymbol{\mu}(x) - \boldsymbol{\mu}(y)\| + \|\boldsymbol{\sigma}(x) - \boldsymbol{\sigma}(y)\| \leq k_0|x - y|, \tag{8}$$

$$\|\boldsymbol{\mu}(x)\|^2 + \|\boldsymbol{\sigma}(x)\|^2 \leq k_0^2(1 + x^2),$$

for any $x, y \in \mathbb{R}$. Also, with $\mathbf{b}(\cdot) = (b_1(\cdot),\ b_2(\cdot),\ \cdots,\ b_m(\cdot))^T$, assume that

$$|a(x) - a(y)| + \|\mathbf{b}(x) - \mathbf{b}(y)\| \leq k_0|x - y|.$$

***Assumption 2***. Given any time point $t > 0$, there exists a constant $L > 0$ such that $E|\mu_i(r_s)|^{4(q_0+\delta)} \leq L$ and $E|\sigma_{ij}(r_s)|^{4(q_0+\delta)} \leq L$ for any $s \in [t - \eta, t]$ and $1 \leq i, j \leq d$, where $\eta$ is some positive constant, $q_0$ is an integer not less than 1, and $\delta$ is some small positive constant.

***Assumption 3***. The solution $\{f_t\}$ of model (6) is a stationary Markov process and real ergodic. For $t \geq 0$, define the transition operator by:

$$(H_t g)(a) = E(g(f_t)|f_0 = a),\ a \in R,$$

where $g(\cdot)$ is any Borel measurable bounded function on $\mathbb{R}$. Suppose $H_t$ satisfy the $G_2$ condition of Rosenblatt (1970), i.e., there is some $s > 0$ such that

$$|H_s|_2 = \sup_{\{g,\ Eg(X)=0\}} \frac{E^{1/2}(H_s g)^2(X)}{E^{1/2} g^2(X)} \leq \alpha < 1.$$

***Assumption 4***. The conditional density $p_\ell(y|x)$ of $f_{t_{i+\ell}}$ given $f_{t_i}$ is continuous in the arguments $(y, x)$ and is bounded by a constant independent of $\ell$. The time-invariant density function $p(x)$ of the process $f_t$ is bounded and continuous.

***Assumption 5***. The kernel $K(\cdot)$ is a continuously differentiable, symmetric probability density function satisfying

$$\int |x^j K'(x)| dx < \infty,\ j = 0, 1, \cdots, 5, \tag{9}$$

$$\mu_i = \int x^i K(x) dx < \infty,\ i = 0, 1, \cdots, 4, \tag{10}$$

and

$$\nu_0 = \int K^2(x) dx < \infty.$$

Let $\{\mathcal{F}_t\}$ be the augmented filtration defined in Lemma 2 of Appendix. Assumption 1 ensures that there exist continuous, adapted processes $\boldsymbol{X} = \{\boldsymbol{X}_t, \in \mathcal{F}_t;\ 0 \leq t < \infty\}$ and $f = \{f_t \in \mathcal{F}_t;\ 0 \leq t < \infty\}$, which are strong solutions to SDEs (4) and (6) respectively, provided that the initial values $\boldsymbol{X}_0$ and $f_0$ satisfy $E\|\boldsymbol{X}_0\|^2 < \infty$ and $E|f_0|^2 < \infty$, and are independent of Brownian motion $\mathbf{W}$ (see, e.g., Chapter 5, Theorem 2.9 of Karatzas and Shreve, 1991). Assumption 2 indicates that, given any time point $t > 0$, there is a time interval $[t - \eta, t]$ on which the drift and volatility functions have finite $4(q_0 + \delta)$-th moments. Assumption 3 says that $f_t$ is stationary and ergodic and satisfies some mixing condition (see Fan, 2005), which ensures that $f_t$ is Harris recurrent. For the stationarity assumption of $f_t$ to be true, see Hansen and Scheinkman (1995) for conditions. Assumption 4 imposes some constraints on the transition density of $f_t$. Assumption 5 is a regularity condition on the kernel function. For example, the commonly used Gaussian kernel satisfies it.

With the above theoretical framework and assumptions, we will formally demonstrate that the nonparametric estimators using the data localizing in time and in state

are asymptotically jointly normal and independent. This gives a formal theoretical justification and serves as the theoretical foundation for the idea that the time-domain and state-domain nonparametric estimators can be combined to yield a more efficient volatility matrix estimator.

# 2 DIFFUSION MATRIX ESTIMATION USING RECENT INFORMATION

The time-domain method has been extensively studied in the literature. See, for example, Robinson (1997), Härdle *et al.* (2002), Fan, Jiang, Zhang and Zhou (2003), and Mercurio and Spokoiny (2004), among others. A popular time-domain method, the moving average estimator is defined as

$$\widehat{\boldsymbol{\Sigma}}_{MA,t} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y}_{N-i}\boldsymbol{Y}_{N-i}^{T}, \tag{11}$$

where $n$ is the size of the moving window. This estimator ignores the drift component and utilizes $n$ local data points. An extension of the moving average estimator is the exponential smoothing estimator, which is defined as

$$\widehat{\boldsymbol{\Sigma}}_{ES,t} = (1-\lambda)\sum_{i=1}^{\infty} \lambda^{i-1}\boldsymbol{Y}_{N-i}\boldsymbol{Y}_{N-i}^{T}, \tag{12}$$

where $\lambda$ is a smoothing parameter controlling the size of the local neighborhood. RiskMetrics of J.P. Morgan (1996), which is used for measuring the risks of financial assets, recommends $\lambda = 0.94$ and $\lambda = 0.97$ when one uses (12) to forecast the daily and monthly volatility, respectively.

The exponential smoothing estimator (12) is one type of rolling sample variance estimator. See Foster and Nelson (1996) for more information about rolling sample variance estimators. Estimator (12) is also related to the multivariate GARCH model in the literature. Note that when $\Delta$ is very small, the first term on the right hand side of (7) can be ignored. Thus (7) and (12) can be written as

$$\boldsymbol{Y}_i \approx \boldsymbol{\sigma}(f_{t_i})\boldsymbol{\varepsilon}_i,$$

$$\boldsymbol{\Sigma}_{t_i} = (1-\lambda)\boldsymbol{Y}_{i-1}\boldsymbol{Y}_{i-1}^{T} + \lambda\boldsymbol{\Sigma}_{t_{i-1}},$$

where $\mathbf{\Sigma}_{t_i} = \boldsymbol{\sigma}(f_{t_i})\boldsymbol{\sigma}(f_{t_i})^T$, which reminisces the IGARCH model.

The exponential smoothing estimator in (12) is a weighted sum of squared returns prior to time $t$. Since the weight decays exponentially, it essentially uses recent data. To explicitly account for this, we use a slightly modified version:

$$\widehat{\mathbf{\Sigma}}_{T,t} = \frac{1-\lambda}{1-\lambda^n}\sum_{i=1}^{n}\lambda^{i-1}\boldsymbol{Y}_{N-i}\boldsymbol{Y}_{N-i}^T. \tag{13}$$

Here, as in the case of the moving average estimator in (11), $n$ is a smoothing parameter controlling explicitly the window width, and $\lambda$ acts like a kernel weight which may depend on $n$. For example, when $\lambda = 1 - \frac{\tau}{n}$ with $\tau$ a positive constant, besides the normalization factor $\frac{1-\lambda}{1-\lambda^n}$, the first data point $Y_{t-1}$ receives weight 1, while the last point $Y_{t-n}$ receives approximately weight $e^{-\tau}$. In particular, when $\lambda = 1$, it becomes the moving average estimator (11).

Before going into the details, we first introduce some notations and definitions. Let $A = (a_{ij})$ be an $m \times n$ matrix. By $\text{vec}(A)$ we mean the $mn \times 1$ vector formed by stacking the columns of $A$. If $A$ is also symmetric, we vectorize the lower half of $A$ and denote the vector by $\text{vech}(A)$. These notations are consistent with Bandi and Moloche (2004). It is not difficult to verify that there exists a unique $m^2 \times m(m+1)/2$ matrix $D$ with elements 0 and 1, such that

$$P_D\text{vec}(A) = \text{vech}(A),$$

where $P_D = (D^TD)^{-1}D^T$. Another useful definition is the Kronecker product of two matrices $A$ and $B$, which is defined as $A \otimes B = (a_{ij}B)$.

Since the estimator $\widehat{\mathbf{\Sigma}}_{T,t}$ is symmetric, we only need to consider the asymptotic normality of the linear combination of the vector $\text{vech}(\widehat{\mathbf{\Sigma}}_{T,t})$:

$$\widehat{U}_{T,t} \equiv \mathbf{c}^T\text{vech}\widehat{\mathbf{\Sigma}}_{T,t} = \frac{1-\lambda}{1-\lambda^n}\sum_{i=1}^{n}\lambda^{i-1}\sum_{k=1}^{d}\sum_{\ell=1}^{k}c_{k\ell}Y_{N-i}^kY_{N-i}^\ell, \tag{14}$$

where $\mathbf{c} = (c_{1,1},\ c_{2,1},\ c_{2,2},\ c_{3,1},\ \cdots,\ c_{d,d})^T$ is a constant vector.

**Proposition 1** *Under Assumptions 1 and 2, for almost every sample path, we have*

$$\|\boldsymbol{\sigma}(r_s) - \boldsymbol{\sigma}(r_u)\| \le K|s-u|^q, \quad s,u \in [t-\eta, t], \tag{15}$$

*where $q = (2q_0-1)/(4q_0)$, $q_0$ is the integer in Assumption 2, and the coefficient $K$ satisfies $E[K^{4(q_0+\delta)}] < \infty$ with $\delta$ a positive constant.*

***Remark 1***. Proposition 1 shows the continuity of $\boldsymbol{\sigma}(r_s)$ as a function of time $s$, which is the foundation of time-domain estimation. In the proof of Proposition 1, we only used Assumption 2 and the condition $\|\boldsymbol{\sigma}(x) - \boldsymbol{\sigma}(y)\| \leq k_0|x - y|$ with $k_0$ a positive constant. Assumption 1 is made to ensure the existence of a solution to model (5).

**Theorem 1** *Suppose that* $n \to \infty$, $n\Delta^{2q/(2q+1)} \to 0$, *and Assumptions 1 and 2 hold at time* $t$. *If the limit* $\tau = \lim_{n\to\infty} n(1-\lambda)$ *exists, then given* $f_t = x$, *the conditional distribution of* $\text{vech}(\widehat{\boldsymbol{\Sigma}}_{T,t})$ *is asymptotically normal, i.e.,*

$$\sqrt{n}\ \text{vech}(\widehat{\boldsymbol{\Sigma}}_{T,t} - \boldsymbol{\Sigma}(x)) \xrightarrow{D} N\left(0, \frac{\tau(1+e^{\tau})}{(e^{\tau}-1)}\Lambda(x)\right),$$

*where* $\Lambda(x) = P_D^T\{\boldsymbol{\Sigma}(x) \otimes \boldsymbol{\Sigma}(x)\}P_D$.

Note that all data used in the estimator (13) is within $n\Delta$ away from time $t$. According to Proposition 1, the approximation error of (13) is at most of order $O((n\Delta)^q)$, which together with the condition $n\Delta^{2q/(2q+1)} \to 0$ in Theorem 1 guarantees that the bias is of order $o(n^{-1/2})$.

# 3 DIFFUSION MATRIX ESTIMATION USING HISTORICAL INFORMATION

The diffusion matrix in (4) can also be regarded as a nonparametric regression given $f_t = x$. See for example its first order approximation (7). Therefore, it can be estimated by using the historical information via localizing on the state variable $f_t$, as illustrated in Figure 1. The local linear smoother studied in Stanton (1997) will be employed. This technique has several nice properties, such as asymptotic minimax efficiency and design adaptation. Further, it automatically corrects edge effects and facilitates bandwidth selection (Fan and Yao, 2003).

In the construction of the state-domain estimator, we will use the $N-1$ data points right before the current time $t$, i.e., the historical data $\{(f_{t_i},\ \boldsymbol{Y}_i),\ i = 0,\ 1,\ \cdots,\ N-1\}$.

It can be shown that the diffusion matrix has the standard interpretation in terms of infinitesimal conditional moments, that is,

$$E[Y_k^i Y_k^j | f_{t_k} = x_0] = v_{ij}(x_0) + O(\Delta).$$

For a given kernel function[4] $K$ and a bandwidth $h$, the local linear estimator $\hat{\beta}_0^{ij}$ of $v_{ij}(x_0)$ is obtained by minimizing the objective function

$$\sum_{k=0}^{N-1}\{Y_k^i Y_k^j + \beta_0^{ij} + (f_{t_k} - x_0)\beta_1^{ij}\}K_h(f_{t_k} - x_0) \tag{16}$$

over $\beta_0^{ij}$ and $\beta_1^{ij}$. Let

$$W_\ell(x) = \sum_{k=0}^{N-1}(f_{t_k} - x)^\ell K_h(f_{t_k} - x) \tag{17}$$

and

$$w_k(x) = K_h(f_{t_k} - x)\{W_2(x) - (f_{t_k} - x)W_1(x)\}/\{W_0(x)W_2(x) - W_1(x)^2\}. \tag{18}$$

Then the local linear estimator in (16) can be expressed as

$$\widehat{\mathbf{\Sigma}}_{S,t}(x) = \sum_{k=0}^{N-1} w_k(x)\boldsymbol{Y}_k\boldsymbol{Y}_k^T. \tag{19}$$

This estimator depends only on the historical data (horizontal bar in Figure 1), and relies on the structure invariability.

The above weight function $w_k(x)$ is called an "equivalent kernel" in Fan and Yao (2003). Expression (19) reveals that the estimator $\widehat{\mathbf{\Sigma}}_{S,t}(x)$ is very much like a conventional kernel estimator except that the "kernel" $w_k(x)$ depends on the design points and locations.

Before establishing the asymptotic normality of $\widehat{\mathbf{\Sigma}}_{S,t}(x)$, we first investigate the asymptotic property of $W_\ell(x)$.

**Proposition 2** *Suppose* $\Delta \to 0$, $N\Delta \to \infty$, *and* $\frac{1}{h}\sqrt{\Delta \log \Delta^{-1}} = o(1)$. *Under Assumptions 3–5, we have*

$$W_\ell(x) = Nh^\ell\{p(x)\mu_\ell + o_{a.s.}(1)\},\ \ell = 0, 1, 2, 3. \tag{20}$$

The results of Proposition 2 are similar to those in Section 6.3.3 of Fan and Yao (2003, p.237), but the proofs are completely different, as we have high frequency data

[4]The kernel function is a probability density, and the bandwidth is its associated scale parameter. Both of them are used to localize the linear regression around the given point $x_0$. The commonly used kernel functions are the Gaussian density and the Epanechnikov kernel $K(x) = 0.75(1 - x^2)_+$.

here. As sampling interval $\Delta \to 0$, the correlations of the sample $\{f_{t_i}\}$ tend to 1. The high correlation makes their proof fail in our case. To attack this problem, we invoke the local time. The definition and some preliminary results of local time can be found in Revuz and Yor (1999, p.221). For the multifactor situation, the local time generally does not exist. However, by using the occupation time of Bandi and Moloche (2004), our results can be generalized to the multifactor situation.

**Theorem 2** *Suppose* $\Delta \to 0$, $N\Delta \to \infty$, $h = O(N^{-1/5})$, *and* $\frac{1}{h}\sqrt{\Delta \log \Delta^{-1}} = o(1)$. *Moreover, suppose that* $\mathbf{\Sigma}(\cdot)$ *is twice differentiable. Under Assumptions*[5] *3–5, the state-domain estimator has the following asymptotic normality*

$$\sqrt{Nh}\ \mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{S,t}(x) - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\mu_2\mathbf{\Sigma}''(x)\big) \xrightarrow{D} \mathcal{N}(0, 2\nu_0 p(x)^{-1}\Lambda(x)),$$

*where* $\mathbf{\Sigma}''(x)$ *is the matrix whose entries are the second derivatives of the corresponding entries of* $\mathbf{\Sigma}(x)$.

Proposition 2 and Theorem 2 are both studied under the assumption of high frequency data over a long time horizon, i.e., $\Delta \to 0$ and $N\Delta \to \infty$. Various studies under this assumption include Arfi (1998), Gobet (2002), and Fan and Zhang (2003).

# 4 DYNAMIC AGGREGATION OF TIME- AND STATE-DOMAIN ESTIMATORS

In this section, we show that the nonparametric estimators in the time and state domains are asymptotically independent. This allows us to combine these two estimators together to yield a more efficient one.

## 4.1 Asymptotic Normality

The time- and state-domain estimators defined in the previous sections are both driven by the factor process $f_t$. Intuitively, with high probability, most of the data they use

[5]The stationarity condition of $f_t$ in Assumption 3 can be weakened to Harris recurrence. See Bandi and Moloche (2004) for asymptotic normality of local constant estimator under recurrence assumption.

are far apart in time. Since the Markov process $f_t$ is stationary and satisfies some mixing condition (Assumption 3), $f_t$ and $f_s$ are asymptotically independent for $t$ and $s$ far away from each other. Since both the time- and state-domain nonparametric estimators are driven by the same factor process $f_t$, it is reasonable to expect that the two estimators are also asymptotically independent. The following theorem formally shows this result.

**Theorem 3** *Under the conditions of Theorems 1 and 2, conditioning on $f_t = x$, we have*

*(i) asymptotic independence:*

$$\begin{pmatrix} \sqrt{Nh}\ \mathrm{vech}\left(\widehat{\mathbf{\Sigma}}_{S,t} - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\mu_2\mathbf{\Sigma}''(x)\right) \\ \sqrt{n}\ \mathrm{vech}\left(\widehat{\mathbf{\Sigma}}_{T,t} - \mathbf{\Sigma}(x)\right) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \begin{pmatrix} 2\nu_0 p(x)^{-1}\Lambda(x) & \mathbf{0} \\ \mathbf{0} & \frac{\tau(1+e^\tau)}{(e^\tau-1)}\Lambda(x) \end{pmatrix}\right).$$

*(ii) asymptotic normality of the aggregrated estimator $\widehat{\mathbf{\Sigma}}_{A,t}(x)$ in (1):*

$$\sqrt{Nh}\ \mathrm{vech}\Big(\widehat{\mathbf{\Sigma}}_{A,t}(x) - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\omega_t(x)\mu_2\mathbf{\Sigma}''(x)\Big) \xrightarrow{D} \mathcal{N}(0, \Omega(x)),$$

*where* $\Omega(x) = \big(2\omega_t^2(x)\nu_0 p(x)^{-1} + b(1-\omega_t(x))^2\frac{\tau(1+e^\tau)}{(e^\tau-1)}\big)\Lambda(x)$*, provided that* $\lim Nh/n = b$ *for some positive constant $b$ and $h = O(N^{-1/5})$.*

From Theorem 3(i) we can see that the asymptotic covariance matrices of $\widehat{\mathbf{\Sigma}}_{S,t}$ and $\widehat{\mathbf{\Sigma}}_{T,t}$ are proportional to a common matrix $\Lambda(x)$, which is the reason that the optimal dynamic weight $\omega_t(x)$ is independent of the allocation vector $\mathbf{a}$, as mentioned in the Introduction. The same kind of result would hold for multi-factor setting. In fact, by using the occupation time of Bandi and Moloche (2004), one can establish similar result in the multi-factor setting without much extra effort. The main difference would be that the invariant density function $p(x)$ of the single factor process $f_t$ is replaced by the joint invariant density function of the multi-factor processes. So even though the proofs have only been derived with one factor, the substance of our idea is actually broader. Note that the nonparametric estimator in the time domain uses $n$ data points and the nonparametric estimator in the state domain effectively uses the amount $O(Nh)$

of data. The condition $\lim Nh/n = b$ ensures that both estimators effectively use the same amount (order) of data, which avoids the trivial case that either the time domain or the state domain dominates the performance.

### 4.2 Choice of the Dynamic Weight

A natural question is how to choose the dynamic weight $\omega_t(x)$. By Theorem 3(i) and (3), it is easy to see that for any allocation vector $\mathbf{a}$, the asymptotic optimal weight is

$$\omega_t(x) = \frac{b\tau(1+e^\tau)p(x)}{2\nu_0(e^\tau - 1) + b\tau(1+e^\tau)p(x)}, \tag{21}$$

which is independent of $\mathbf{a}$. This choice[6] also optimizes the performance of the aggregated covariance estimator $\widehat{\mathbf{\Sigma}}_{A,t}(x)$. Indeed, by Theorem 3(ii), the asymptotic covariance matrix of $\widehat{\mathbf{\Sigma}}_{A,t}(x)$ is given by $\Omega(x)$. It depends on the weight through the coefficient

$$\psi_t(x) \equiv 2\omega_t^2(x)\nu_0 p(x)^{-1} + b(1-\omega_t(x))^2 \frac{\tau(1+e^\tau)}{(e^\tau - 1)},$$

which is a quadratic function, and attains its minimum at (21).

When $0 < b < \infty$, the effective sample sizes in the time and state domains are comparable. Hence, neither the time-domain nor the-state domain estimator dominates. Therefore, by aggregating the time- and state-domain estimators, we obtain an optimal reduction of asymptotic variance. The biases of the aggregated estimator are indirectly controlled, when the optimal smoothing is conducted for both time- and state-domain estimators so that their biases and variances are already traded off before aggregation.

Note that at time $t$, the optimal weight $\omega_t(x)$ depends on the current value of the factor process $f$ through the density function $p(x)$. This is consistent with our common sense. When $f$ is low or high, $p(x)$ and consequently, the optimal weight are approximately zero. In this case, the main contribution to the aggregated estimator comes from the time-domain estimator. When $f$ is well in middle of its state space, say near its unconditional mathematical expectation, the state-domain estimator tends to dominate the aggregated estimator.

[6]The optimal choice of weight is proportional to the effective number of data points used for the state-domain and time-domain smoothing. It always outperforms the choice with $\omega_t = 1$ (state-domain estimator) or $\omega_t = 0$ (time-domain estimator).

In practice, the density function $p(x)$ is unknown and should be estimated. There are lots of existing methods to do this, such as the kernel density estimator and the local time density estimator (see Aït-Sahalia, 1996; and Dalalyan and Kutoyants, 2003).

# 5 NUMERICAL ANALYSIS

To evaluate the aggregated estimator, we compare it with the time-domain estimator and the state-domain estimator. For the time-domain estimation, we apply the exponential smoothing[7] with $\lambda = 0.94$. For the state-domain estimation, we choose one yield process as the "factor," and then use it to estimate the volatility matrix. The Epanechnikov kernel is used with the bandwidth $h$ chosen by generalized cross validation method (see Fan and Yao, 2003). To choose the optimal weight $\omega_t(x)$, we estimate the density function $p(x)$ by the kernel density estimator (see Aït-Sahalia, 1996).

The following three measures are employed to assess the performance of different methods for estimating the diffusion matrix. The first two can only be used in simulation, and the last one can be used in both simulation and real data analysis.

**Measure 1.** The entropy loss is given by

$$l_1(\boldsymbol{\Sigma}_t,\ \widehat{\boldsymbol{\Sigma}}_t) = \mathrm{tr}(\boldsymbol{\Sigma}_t^{-1}\widehat{\boldsymbol{\Sigma}}_t) - \log|\boldsymbol{\Sigma}_t^{-1}\widehat{\boldsymbol{\Sigma}}_t| - \dim(\boldsymbol{\Sigma}_t).$$

**Measure 2.** The quadratic loss is defined as

$$l_2(\boldsymbol{\Sigma}_t,\ \widehat{\boldsymbol{\Sigma}}_t) = \mathrm{tr}\big(\widehat{\boldsymbol{\Sigma}}_t - \boldsymbol{\Sigma}_t\big)^2.$$

**Measure 3.** The prediction error (PE) is computed as

$$\mathrm{PE}(\widehat{\boldsymbol{\Sigma}}_t) = \frac{1}{m}\sum_{i=T+1}^{T+m} \mathrm{tr}\big(\boldsymbol{Y}_i\boldsymbol{Y}_i^T - \widehat{\boldsymbol{\Sigma}}_{t_i}\big)^2 \tag{22}$$

for an out-sample of size $m$. The expected value can be decomposed as

$$E[\mathrm{PE}(\widehat{\boldsymbol{\Sigma}}_t)] = \frac{1}{m}\sum_{i=T+1}^{T+m} E[\mathrm{tr}\left(\boldsymbol{Y}_i\boldsymbol{Y}_i^T - \boldsymbol{\Sigma}_{t_i}\right)^2] + \frac{1}{m}\sum_{i=T+1}^{T+m} E[\mathrm{tr}\big(\boldsymbol{\Sigma}_{t_i} - \widehat{\boldsymbol{\Sigma}}_{t_i}\big)^2].$$

[7]The choice comes from the recommendation of the RiskMetrics of J.P. Morgan. The parameter $\lambda$ can also be chosen automatically by data by using the prediction error as in Fan, Jiang, Zhang and Zhou (2003). Since we compare the relative performance between the time-domain estimator and the aggregated estimator, we opt for this simple choice. The results do not expect to change much when a data-driven technique is used.

Note that the second item reflects the effectiveness of the estimated diffusion matrix, while the first term is the size of the stochastic error, independent of the estimators. The first term is usually an order of magnitude larger than the second term. Thus, a small improvement in PE means a substantial improvement in estimated volatility. This will also be clearly demonstrated in our simulation study (see Figure 4).

**Measure 4.** Adaptive prediction error (APE).

As seen above, the dominant part of the PE is the stochastic error; however, what we really care about is the estimation error. To reduce the stochastic error in (22), we define the following adaptive prediction error:

$$\mathrm{APE}(\widehat{\boldsymbol{\Sigma}}_t) = \frac{1}{m}\sum_{i=T+1}^{T+m} \mathrm{tr}\big(\frac{1}{2k+1}\sum_{j=i-k}^{i+k} \boldsymbol{Y}_j \boldsymbol{Y}_j^T - \widehat{\boldsymbol{\Sigma}}_{t_i}\big)^2, \tag{23}$$

where $k$ is a nonnegative integer. The basic idea is to average out the stochastic errors first before computing square losses, but this creates bias when $k$ is large. When $k = 0$, the APE reduces to the PE defined in (22).

## 5.1 Simulation

We use an essentially affine market price of risk specifications in Duffee (2002) to simulate bond yields, and hence to obtain simulated multivariate time series. Essentially affine model is the multivariate extension of the square-root process. It has been proved useful in forecasting future yields (see Duffee, 2002). Cheridito, Filipović and Kimmel (2005) investigate the essentially affine model with one, two, and three state variables, and give estimates of the parameters. We use their one state variable model to conduct the simulations.

The one state variable affine term structure model assumes that the instantaneous nominal interest rate $r_t$ is given by

$$r_t = d_0 + d_1 s_t,$$

where $d_0$ and $d_1$ are scalars, and $s_t$ is a scalar state variable. The evolution of the state variable $s_t$ under the the risk-neutral measure $Q$ is assumed to be

$$ds_t = \big(a_1^Q + b_{11}^Q s_t\big)dt + \sqrt{s_t} dW_t^Q. \tag{24}$$

This is the well-known Cox-Ingersoll-Ross (CIR) model.

Let $P(t,\tau)$ be the time-$t$ price of a zero-coupon bond maturing at $t+\tau$. Under the affine term structure and the assumption of no arbitrage, Duffie and Kan (1996) show that the bond price admits the form

$$P(t,\tau) = E_t^Q \exp(-\int_t^{t+\tau} r_u du) = \exp[A(\tau) - B(\tau)s_t], \tag{25}$$

where $A(\tau)$ and $B(\tau)$ are both scalar functions satisfying the following ordinary differential equations (ODEs)

$$\frac{dA(\tau)}{d\tau} = -a_1^Q B(\tau) - d_0 \text{ and } \frac{dB(\tau)}{d\tau} = b_{11}^Q B(\tau) - \frac{1}{2}B^2(\tau) + d_1. \tag{26}$$

Thus, the bond's yield

$$y(s_t,\tau) = -\frac{1}{\tau}\log P(t,\tau) = \frac{1}{\tau}[-A(\tau) + B(\tau)s_t] \tag{27}$$

is affine in the state variable $s_t$.

We use the above model to simulate 5 zero-coupon bond yield processes with maturities 1 month, 2 years, 4 years, 6 years, and 8 years. Since there is only one state variable $s_t$, the bond yields of different maturities are perfectly linearly related, as shown in (27), which is an unrealistic artifact of the model. To attenuate this dilemma, Cherito *et al.* (2005) assume that only the 1-month yield process is observed without error, while other yields are contaminated with i.i.d. multivariate Gaussian errors with mean zero and unknown covariance matrix. They estimate the unknown parameters from the yields of zero-coupon bonds extracted from the US Treasury security prices from January 1972 to December 2002. The estimated parameters are $a_1^Q = 0.5$, $b_{11}^Q = -0.0137$, $d_0 = 0.0110$, and $d_1 = 0.0074$. The standard deviations of the Gaussian errors are estimated as $\sigma_1 = 0.0119$, $\sigma_2 = 0.0144$, $\sigma_3 = 0.0155$, and $\sigma_4 = 0.0159$ for the yields of 2-, 4-, 6-, and 8-year bonds, respectively. The associated correlation coefficients are estimated as $\rho_{12} = 0.9727$, $\rho_{13} = 0.9511$, $\rho_{14} = 0.9371$, $\rho_{23} = 0.9950$, $\rho_{24} = 0.9877$, and $\rho_{34} = 0.9978$.

Figure 2 here.

In the simulation, we set the the parameter values to be the above estimated values from Cherito *et al.* (2005). We first generate discrete samples of the state variable $s_t$

from diffusion process (24). Then we solve ODEs in (26) numerically. Figure 2 shows the solution to (26). After that, we obtain the ideal yield processes by using (27) with maturities 1 month, 2 years, 4 years, 6 years, and 8 years. Finally, we add the i.i.d. 4-variate normal errors to the last 4 ideal yield processes to obtain the observed bond processes with these maturities[8].

To generate the sample path of $s_t$, we use the transition density property of the process. That is, given $s_t = x$, the variable $2cs_{t+\Delta}$ has a noncentral chi-squared distribution with degrees of freedom $4a_{11}^Q$ and noncentrality parameter $2cxe^{b_{11}^Q\Delta}$, where $c = \frac{2b_{11}^Q}{\exp(b_{11}^Q\Delta)-1}$. The initial value of $s_0$ is generated from the invariant distribution of $s_t$, which is gamma distribution with density $p(y) = \frac{\omega^\nu}{\Gamma(\nu)}y^{\nu-1}e^{-\omega y}$, where $\nu = 2a_{11}^Q$ and $\omega = -2b_{11}^Q$.

We simulate 500 series of 1200 observations of weekly data with $\Delta = 1/52$ for the yields of five zero-coupon bonds with maturities 1 month, 2 years, 4 years, 6 years, and 8 years, respectively. For each simulated series, we set the last 150 observations as the out-sample data. For time $t$ out-sample data point, the time-domain estimator is based on the past $n = 104$ (two years)[9] observations, i.e., observations from $t-104$ to $t-1$; and the state-domain estimator is based on the 1050 data points right before the current time, i.e., the data points from time $t-1050$ to $t-1$. The first yields process (1-month) is used as the factor for state-domain estimation.

As pointed out in Section 1, the conditional covariance matrix of the multivariate diffusion can be approximated by the diffusion matrix times the sampling interval $\Delta$. Hence, we first obtain estimates of the diffusion matrix, and then convert them into the conditional covariance matrix estimates. The theoretical value of the conditional variance of $s_t$ is given by Duffee (2002). Since the bond yields are linear regression models of the state variable (see (27) with Gaussian errors), the true (theoretical) value of the conditional covariance matrix of the bond yields can be easily obtained.

[8]Here we add normal noise to make the model more realistic. Our method performs even better without noise. Since the noise vectors are i.i.d. across time and the standard deviations are small, adding them to the original time series does not change the whole structure. Hence, our theory can carry through under contamination.

[9]With $\lambda = 0.94$, the last data point used in the time domain has an extra weight $0.94^{104} \approx 0.0016$, which is very small. Hence, we essentially include all the effective data points.

By comparing the estimated conditional covariance matrix to its theoretical value, the performance of our estimation procedures is evaluated.

Figure 3 here.

Figure 3 depicts the averages and standard deviations of the entropy and quadratic losses of time-domain, state-domain, and aggregated estimators. It shows unambiguously that the aggregated method always has the smallest averages and standard deviations across 500 simulations for both the entropy loss and quadratic loss. Figures 4(a) and 4(b) summarize the distributions of the average losses over 150 out-samples forecasting across the 500 simulations. The results are consistent with those in Figure 3. On the other hand, if the PE in (22) with $m = 150$ is used, the distributions look quite different, which is demonstrated in Figure 4(c). It shows clearly that even though there are huge efficiency improvements in estimating the volatility matrix by using the aggregated method, the improvements are masked by stochastic errors which are an order of magnitude larger than the estimation errors. The average prediction errors over 500 simulations are $1.850 \times 10^{-2}$, $1.825 \times 10^{-2}$, and $1.846 \times 10^{-2}$ for the time-domain, the aggregated, and the state-domain estimators, respectively. This demonstrates that a small improvement in PE means a huge improvement in the estimation of the volatility matrix. This effect is more illuminatingly illustrated in Figure 4(d) where each point represents a simulation. The $x$-axis represents the ratios of the averages of 150 quadratic losses for the time-domain estimator and the state-domain estimator to those for the aggregated estimator, whereas the $y$-axis is the ratios of the PEs for the time-domain estimator and the state-domain estimator to those for the aggregated estimator. The $x$-coordinates are mostly greater than 1, showing the improved efficiency of the aggregated estimation. On the other hand, the improved efficiency is masked by stochastic errors, resulting in the $y$-coordinate spreading around the line $y = 1$.

Figure 4 here.

We have proved theoretically that nonparametric estimators based on time-domain smoothing and state-domain smoothing are asymptotically independent. To verify this, we compute their correlation coefficients. Since both estimators are matrices, for

a given portfolio allocation vector $\mathbf{a}$, we compute the correlation of the two estimators $\mathbf{a}^T\widehat{\mathbf{\Sigma}}_{T,t}\mathbf{a}$ and $\mathbf{a}^T\widehat{\mathbf{\Sigma}}_{S,t}\mathbf{a}$ across 500 simulations at each given time $t$ in the out-sample. Figure 5 presents the correlation coefficients for $\mathbf{a} = (0.2,\ 0.2,\ 0.2,\ 0.2,\ 0.2)^T$. Most of the correlations are below 0.1, which strongly supports our theoretical result. We also include the 95% confidence intervals based on the Fisher transformation in the same graph (the two dashed curves). A large amount of these confidence intervals contain 0. The two straight lines in the plot indicate the acceptance region for testing the null hypothesis that the correlation coefficients are zero at the significance level 5%. Most of these null hypotheses are accepted or nearly accepted. In fact, we conducted experiments on the same simulations with larger sample sizes, and found that as the sample size increases, the absolute values of the correlation coefficients decrease to 0.

Figure 5 here.

## 5.2 Empirical Studies

In this section, we apply the aggregated method to two sets of financial data. Our aim is to examine whether our approach still outperforms the time-domain and state-domain nonparametric estimators in real applications.

### 5.2.1 Treasury Bonds

We consider the weekly returns of five treasury bonds with maturities 3 months, 2 years, 5 years, 7 years, 10 years, and 30 years. We set the last 150 observations, which run from April 9, 1999 to February 15, 2002, as the out-sample data. For each observation from the out-sample data, we use the past 104 observations (2 years) with $\lambda = 0.94$ to obtain the time-domain estimator, and the state-domain estimate is based on the past 900 data points. The prediction error (Measure 3) and adaptive prediction error (Measure 4) are used to assess the performance of the three estimators: the time-domain estimator, the state-domain estimator, and the aggregated estimator. The results are reported in Table 1. From the table, we see that the aggregated estimator outperforms significantly the other two estimators.

For comparison, the results from the simulated data are also reported. Even through there is only a small improvement in PE for simulated data, as evidenced in Section

4.1, there is a huge improvement in the precision of estimating $\mathbf{\Sigma}_t$ in terms of entropy loss (measure 1) and quadratic loss (measure 2). Hence, with the improvement of the PE in the bond price by the aggregrated method, we would expect to have a huge improvement on the precision of the estimation of covariance, which is of primary interest in financial engineering.

### 5.2.2 Exchange Rate

We analyze the weekly exchange rates of five foreign currencies with US dollars from September 6, 1985 to August 19, 2005. The five foreign currencies are the Canadian Dollar, Australian Dollar, Europe Euro[10], UK British Pound, and Switzerland Franc. The length of the time series is 1042. The exchange rates from December 6, 2002 to August 19, 2005, which are of length 142, are regarded as out-sample data, and the estimation procedures are the same as before, i.e., for each out-sample observation, the last 104 data points with $\lambda = 0.94$ are set to construct the time-domain estimator, the 900 data points before the current time are used to construct state-domain estimator, and then roll over. The results, based on the PE and APE defined in Section 4, are also summarized in Table 1. They demonstrate clearly that the aggregated estimator outperforms the time-domain and state-domain estimators.

Using again the simulated data for calibration, as argued at the end of Section 4.2.1, we would reasonably expect that the covariance matrix estimated by the aggregated method outperforms significantly both the matrices estimated by either the time- or state-domain method alone.

Table 1 here.

# 6 DISCUSSIONS

We have proposed an aggregated method to combine the information from the time domain and state domain in multivariate volatility estimation. To overcome the *curse*

[10]Europe used several common currencies prior to the introduction of the Euro. The European Currency Unit (ECU) was used from January 1, 1979 to January 1, 1999, when the Euro replaced the European Currency Unit at par.

*of dimensionality*, we proposed a "factor" modeling strategy. The performance comparisons are studied both theoretically and empirically. We have shown that the proposed aggregated method is more efficient than the estimators based only on recent history or remote history. Our simulation and empirical studies have also revealed that proper use of information from both the time domain and the state domain makes volatility matrix estimate more accurate. Our method exploits the continuity in the time domain and stationarity in the state domain. It can also be applied to situations where these two conditions hold approximately.

Our study has also revealed another potentially important application of our method. It allows us to test the stationarity of diffusion processes. When time-domain estimates differ substantially from those of the state domain, it is an indication that the processes is not stationary. Since the time-domain and state-domain nonparametric estimators are asymptotically independent and normal, formal tests can be formed. Further study on this topic is beyond the scope of this paper.

# APPENDIX: PROOFS

## A.1 Proof of Proposition 1

In all the proofs below, we use $M$ to denote a generic constant.

First, we show that the process $\{f_t\}$ is locally Hölder continuous with order $q = (2q_0 - 1)/(4q_0)$ and coefficient $K_1$ satisfying $E[K_1^{4(q_0+\delta)}] < \infty$, i.e.

$$|f_s - f_u| \leq K_1|s-u|^q, \quad s, u \in [t-\eta, t], \tag{A.1}$$

where $\eta$ is a positive constant. Note that

$$\begin{aligned} E|f_u - f_s|^{4(q_0+\delta)} &\leq ME\Big|\int_s^u a(f_v)dv\Big|^{4(q_0+\delta)} + ME\Big|\int_s^u \sum_j b_j(f_v)dW_v^j\Big|^{4(q_0+\delta)} \\ &\equiv (I) + (II). \end{aligned} \tag{A.2}$$

Then by Jensen's inequality and Assumption 2, we have

$$(I) \leq M(u-s)^{4(q_0+\delta)-1}\int_s^u E|a(f_v)|^{4(q_0+\delta)}dv \leq M(u-s)^{4(q_0+\delta)}. \tag{A.3}$$

On the other hand, applying martingale moment inequalities (see, e.g. Karatzas and Shreve (1991), Section 3.3.D, p.163), Jensen's inequality, and Assumption 2 gives

$$(II) \leq ME\Big(\int_s^u \sum_j b_j^2(f_v)dv\Big)^{2(q_0+\delta)} \leq M(u-s)^{2(q_0+\delta)-1}\int_s^u \sum_j E|b_j(f_v)|^{4(q_0+\delta)}dv \tag{A.4}$$

$$\leq M(u-s)^{2(q_0+\delta)}.$$

Combining (A.2), (A.3) and (A.4) together leads to

$$E|f_u - f_s|^{4(q_0+\delta)} \leq M(u-s)^{2(q_0+\delta)}.$$

Thus by Theorem 2.1 of Revuz and Yor (1999, Page 26), we have

$$E\big[\big(\sup_{s\neq u}\{|f_s - f_u|/|s-u|^{\alpha}\}\big)^{4(q_0+\delta)}\big] < \infty \tag{A.5}$$

for any $\alpha \in [0, \frac{2(q_0+\delta)-1}{4(q_0+\delta)})$. Let $\alpha = \frac{2q_0-1}{4q_0}$ and $K_1 = \sup_{s\neq u}\{|f_s - f_u|/|s-u|^{\frac{2q_0-1}{4q_0}}\}$. Then $E[K_1^{4(q_0+\delta)} < \infty]$, and inequality (A.1) holds.

Second, by (8) we have

$$\|\boldsymbol{\sigma}(f_s) - \boldsymbol{\sigma}(f_u)\| \leq k_0|f_s - f_u|.$$

This together with (A.1) shows that

$$\|\boldsymbol{\sigma}(f_s) - \boldsymbol{\sigma}(f_u)\| \leq k_0K_1|s-u|^q \equiv K|s-u|^q.$$

Hence, $E[K^{4(q_0+\delta)}] \leq ME[K_1^{4(q_0+\delta)}] < \infty$. ■

## A.2 Proof of Theorem 1

**Proof**. At time $s$, for fixed $k$, $\ell$, and $i$, define $Z_{i,s}^{k,\ell} = (X_s^k - X_{t_i}^k)(X_s^\ell - X_{t_i}^\ell)$. Applying

Ito's formula to $Z_{i,s}^{k,\ell}$ results in

$$\begin{aligned}
dZ_{i,s}^{k,\ell} =&(X_s^k - X_{t_i}^k)dX_s^\ell + (X_s^\ell - X_{t_i}^\ell)dX_s^k + \sum_{j=1}^m \sigma_{kj}(f_s)\sigma_{\ell j}(f_s)ds \\
=&\left[(X_s^k - X_{t_i}^k)\mu_\ell(f_s) + (X_s^\ell - X_{t_i}^\ell)\mu_k(f_s)\right]ds \\
&+ [\int_{t_i}^s \mathbf{e}_k^T\boldsymbol{\mu}(f_u)du\mathbf{e}_\ell^T\boldsymbol{\sigma}(f_s) + \int_{t_i}^s \mathbf{e}_\ell^T\boldsymbol{\mu}(f_u)du\mathbf{e}_k^T\boldsymbol{\sigma}(f_s)]d\mathbf{W}_s \\
&+ [\int_{t_i}^s \mathbf{e}_k^T\boldsymbol{\sigma}(f_u)d\mathbf{W}_u\mathbf{e}_\ell^T\boldsymbol{\sigma}(f_s) + \int_{t_i}^s \mathbf{e}_\ell^T\boldsymbol{\sigma}(f_u)d\mathbf{W}_u\mathbf{e}_k^T\boldsymbol{\sigma}(f_s)]d\mathbf{W}_s \\
&+ \sum_{j=1}^m \sigma_{kj}(f_s)\sigma_{\ell j}(f_s)ds.
\end{aligned}$$

Hence, $Y_i^k Y_i^\ell$ can be decomposed as

$$Y_i^k Y_i^\ell = \Delta^{-1} Z_{i,t_{i+1}}^{k,\ell} \equiv a_i^{k,\ell} + b_i^{k,\ell} + \bar{v}_i^{k,\ell},$$

where

$$\begin{aligned}
a_i^{k,\ell} =&\Delta^{-1}\int_{t_i}^{t_{i+1}} [(X_s^k - X_{ti}^k)\mu_\ell(f_s) + (X_s^\ell - X_{t_i}^\ell)\mu_k(f_s)]ds \\
&+ \Delta^{-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^s [\mathbf{e}_k^T\boldsymbol{\mu}(f_u)du\mathbf{e}_\ell^T\boldsymbol{\sigma}(f_s) + \mathbf{e}_\ell^T\boldsymbol{\mu}(f_u)du\mathbf{e}_k^T\boldsymbol{\sigma}(f_s)]d\mathbf{W}_s,
\end{aligned}$$

$$b_i^{k,\ell} = \Delta^{-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^s [\mathbf{e}_k^T\boldsymbol{\sigma}(f_u)d\mathbf{W}_u\mathbf{e}_\ell^T\boldsymbol{\sigma}(f_s) + \mathbf{e}_\ell^T\boldsymbol{\sigma}(f_u)d\mathbf{W}_u\mathbf{e}_k^T\boldsymbol{\sigma}(f_s)]d\mathbf{W}_s$$

and

$$c_i^{k,\ell} = \Delta^{-1}\int_{t_i}^{t_{i+1}} \sum_{j=1}^m \sigma_{kj}(f_s)\sigma_{\ell j}(f_s)ds.$$

Correspondingly, (14) has the following decomposition

$$\begin{aligned}
\widehat{U}_{T,t} =& \frac{1-\lambda}{1-\lambda^n}\sum_{i=1}^n \lambda^{i-1}\sum_{\ell\le k} c_{k\ell}a_{N-i}^{k,\ell} + \frac{1-\lambda}{1-\lambda^n}\sum_{i=1}^n \lambda^{i-1}\sum_{\ell\le k} c_{k\ell}b_{N-i}^{k,\ell} \\
&+ \frac{1-\lambda}{1-\lambda^n}\sum_{i=1}^n \lambda^{i-1}\sum_{\ell\le k}^k c_{k\ell}\bar{v}_{N-i}^{k,\ell} \\
\equiv& A_{n,\Delta} + B_{n,\Delta} + V_{n,\Delta}.
\end{aligned} \tag{A.6}$$

Therefore, Slutsky's lemma, together with Lemmas 1–3 below, leads to the conclusions of Theorem 1 immediately. ■

**Lemma 1** *Under Assumption 1, as $n \to \infty$, $n\Delta \to 0$, and $n(1-\lambda) \to \tau$, we have*

$$EA_{n,\Delta}^2 = O(\Delta), \tag{A.7}$$

*where $A_{n,\Delta} = \frac{1-\lambda}{1-\lambda^n} \sum_{i=1}^n \lambda^{i-1} \sum_{\ell \le k} c_{k\ell} a_{N-i}^{k,\ell}$, as defined in (A.6).*

***Proof***. First, note that

$$\begin{aligned} E(a_i^{k,\ell})^2 \le & 2E\big(\Delta^{-1} \int_{t_i}^{t_{i+1}} [(X_s^k - X_{t_i}^k)\mu_\ell(f_s) + (X_s^\ell - X_{t_i}^\ell)\mu_k(f_s)]ds\big)^2 \\ & + 2E\big(\Delta^{-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^s [\mathbf{e}_k^T \boldsymbol{\mu}(f_u) du \mathbf{e}_\ell^T \boldsymbol{\sigma}(f_s) + \mathbf{e}_\ell^T \boldsymbol{\mu}(f_u) du \mathbf{e}_k^T \boldsymbol{\sigma}(f_s)] d\mathbf{W}_s\big)^2 \\ \equiv & I_1(\Delta) + I_2(\Delta). \end{aligned} \tag{A.8}$$

Applying Jensen's inequality and Hölder's inequality (Propostion 1), we obtain

$$\begin{aligned} I_1(\Delta) \le & M\Delta^{-1} \int_{t_i}^{t_{i+1}} E\left[(X_s^k - X_{t_i}^k)\mu_\ell(f_s) + (X_s^\ell - X_{t_i}^\ell)\mu_k(f_s)\right]^2 ds \\ \le & M\Delta^{-1} \int_{t_i}^{t_{i+1}} \left\{\big(E(X_s^k - X_{t_i}^k)^4 E[\mu_\ell(f_s)]^4\big)^{1/2} + \big(E(X_s^\ell - X_{t_i}^\ell)^4 E[\mu_k(f_s)]^4\big)^{1/2}\right\} ds. \end{aligned} \tag{A.9}$$

Since an application of Jensen's inequality, martingale moments inequalities and Assumption 2 results in

$$\begin{aligned} E(X_s^\ell - X_{t_i}^\ell)^4 &\le M\big(E\big[\int_{t_i}^s \mu_\ell(f_u)du\big]^4 + \sum_{j=1}^m E\big[\int_{t_i}^s \sigma_{\ell j}(f_u)dW_u^j\big]^4\big) \\ &\le M\big((s-t_i)^3 \int_{t_i}^s E[\mu_\ell(f_u)]^4 du + \sum_{j=1}^m M(s-t_i) \int_{t_i}^s E[\sigma_{\ell j}(f_u)]^4 du\big) \\ &\le M(s-t_i)^2, \end{aligned}$$

we see that (A.9) can be bounded as

$$I_1(\Delta) \le M\Delta. \tag{A.10}$$

We now consider the second term $I_2(\Delta)$ in (A.8). By stochastic calculus and Jensen's inequality, we have

$$\begin{aligned} I_2(\Delta) &= 2\int_{t_i}^{t_{i+1}} \sum_{j=1}^m E\Big(\Delta^{-1} \int_{t_i}^s [\mu_k(f_u)\sigma_{\ell j}(f_s) + \mu_\ell(f_u)\sigma_{kj}(f_s)]du\Big)^2 ds \\ &\le M\Delta^{-1} \int_{t_i}^{t_{i+1}} \sum_{j=1}^m \int_{t_i}^s E[\mu_k(f_u)\sigma_{\ell j}(f_s) + \mu_\ell(f_u)\sigma_{kj}(f_s)]^2 du ds \\ &= O(\Delta). \end{aligned}$$

This together with (A.10) leads to $E(a_i^{k,\ell})^2 = O(\Delta)$. Therefore, by the Cauchy-Schwarz inequality and the assumption that $\lim_n(1-\lambda)$ exists,

$$EA_{n,\Delta}^2 \leq Mn\left(\frac{1-\lambda}{1-\lambda^n}\right)^2 \sum_{i=1}^{n} \lambda^{2(i-1)} \sum_{\ell \leq k} c_{k\ell}^2 E(a_{N-i}^{k,\ell})^2 = O(\Delta),$$

which concludes the proof. ■

**Lemma 2** *Under Assumptions 1 and 2, as $n \to \infty$, $n\Delta^q \to 0$ and $n(1-\lambda) \to \tau$, we have*

$$\sqrt{n} B_{n,\Delta} \xrightarrow{D} Z_{\mathbf{C}},$$

*where $B_{n,\Delta}$ is defined in (A.6) and the random variable $Z_{\mathbf{C}}$ is defined in Theorem 1.*

***Proof***. We will decompose $B_{n,\Delta}$ into two parts and prove that the first part is asymptotically negligible and the second part has some asymptotic distribution.

Note that $b_i^{k,\ell}$ can be decomposed as

$$b_i^{k,\ell} = \mathcal{B}_i^{k,\ell} + \mathcal{C}_i^{k,\ell}, \tag{A.11}$$

where

$$\mathcal{B}_i^{k,\ell} = \Delta^{-1} \sum_{j,p} (\sigma_{kj}(f_{t_0})\sigma_{\ell p}(f_{t_0}) + \sigma_{kp}(f_{t_0})\sigma_{\ell j}(f_{t_0})) \int_{t_i}^{t_{i+1}} (W_s^j - W_{t_i}^j) dW_s^p$$

and

$$\mathcal{C}_i^{k,\ell} = \Delta^{-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} [\mathbf{e}_k^T(\boldsymbol{\sigma}(f_u) - \boldsymbol{\sigma}(f_{t_0}))d\mathbf{W}_u \mathbf{e}_\ell^T \boldsymbol{\sigma}(f_s) + \mathbf{e}_k^T \boldsymbol{\sigma}(f_u) d\mathbf{W}_u \mathbf{e}_\ell^T (\boldsymbol{\sigma}(f_s) - \boldsymbol{\sigma}(f_{t_0}))] d\mathbf{W}_s,$$

where $\mathbf{e}_k$ is the unit vector with $k$th entry 1 and all other entries 0. Correspondingly, $B_{n,\Delta}$ is decomposed as

$$B_{n,\Delta} = \frac{1-\lambda}{1-\lambda^n} \sum_{k \leq \ell} c_{k\ell} \sum \lambda^{i-1} \mathcal{B}_{N-i}^{k,\ell} + \frac{1-\lambda}{1-\lambda^n} \sum_{k \leq \ell} c_{k\ell} \sum \lambda^{i-1} \mathcal{C}_{N-i}^{k,\ell} \equiv \mathcal{B} + \mathcal{C}.$$

First, we show that $\sqrt{n}\mathcal{C}$ is asymptotically negligible. To this end, note that by stochastic calculus and the triangular inequality, we have

$$\begin{aligned}
E(\mathcal{C}_i^{k,\ell})^2 \leq & \Delta^{-2} \int_{t_i}^{t_{i+1}} \sum_{j=1}^{m} E\Big( \int_{t_i}^{s} \mathbf{e}_k^T(\boldsymbol{\sigma}(f_u) - \boldsymbol{\sigma}(f_{t_0}))d\mathbf{W}_u \sigma_{\ell j}(f_s) \Big)^2 ds \\
& + \Delta^{-2} \int_{t_i}^{t_{i+1}} \sum_{j=1}^{m} E\Big( \int_{t_i}^{s} \mathbf{e}_k^T \boldsymbol{\sigma}(f_u) d\mathbf{W}_u (\sigma_{\ell j}(f_s) - \sigma_{\ell j}(f_{t_0})) \Big)^2 ds \\
\equiv & \Delta^{-2} \int_{t_i}^{t_{i+1}} \sum_{j=1}^{m} I_1^{(j)}(\Delta) ds + \Delta^{-2} \int_{t_i}^{t_{i+1}} \sum_{j=1}^{m} I_1^{(j)}(\Delta) ds.
\end{aligned}$$

Applying Hölder's inequality yields

$$I_1^{(j)}(\Delta) \leq \Big(E\big(\int_{t_i}^{s} \mathbf{e}_k^T(\boldsymbol{\sigma}(f_u) - \boldsymbol{\sigma}(f_{t_0}))d\mathbf{W}_u\big)^4 E(\sigma_{\ell j}(f_s))^4\Big)^{1/2}, \tag{A.12}$$

and then by martingale moment inequalities and (15) we obtain

$$\begin{aligned} E\Big(\int_{t_i}^{s} \mathbf{e}_k^T(\boldsymbol{\sigma}(f_u) - \boldsymbol{\sigma}(f_{t_0}))d\mathbf{W}_u\Big)^4 \leq & O(1)E\Big(\int_{t_i}^{s}\sum_{j=1}^{m}(\sigma_{kj}(f_u) - \sigma_{kj}(f_{t_0}))^2 du\Big)^2 \\ \leq & O\left((n\Delta + \Delta)^{4q}\Delta^2\right). \end{aligned}$$

Hence, we can bound (A.12) as

$$I_1^{(j)}(\Delta) \leq O\left((n\Delta)^{2q}\Delta\right). \tag{A.13}$$

Next we consider $I_2^{(j)}(\Delta)$. Similarly, by Hölder's inequalities, martingale moments inequalities, and (15) we have

$$\begin{aligned} I_2^{(j)}(\Delta) &\leq \Big(E\big(\int_{t_i}^{s} \mathbf{e}_k^T\boldsymbol{\sigma}(f_u)d\mathbf{W}_u\big)^4 E(\sigma_{\ell j}(f_s) - \sigma_{\ell j}(f_{t_0}))^4\Big)^{1/2} \\ &\leq O(1)\Big(E[\int_{t_i}^{s}\sum_{j=1}^{m}\sigma_{kj}^2(f_u)du]^2(n\Delta + \Delta)^{4q}EK^4\Big)^{1/2} \\ &\leq O\left((n\Delta)^{2q}\Delta\right). \end{aligned}$$

This together with (A.13) implies that

$$E(\mathcal{C}_i^{k,\ell})^2 = O\left((n\Delta)^{2q}\right).$$

Hence, it follows that

$$E\left(\sqrt{n}\mathcal{C}\right)^2 = O((n\Delta)^{2q}), \tag{A.14}$$

which means that $\sqrt{n}\mathcal{C}$ is asymptotically negligible.

Next, we consider the term $\sqrt{n}\mathcal{B}$. We first define the augmented filtration $\mathcal{F}_t$. Let $(\Omega, \mathcal{F}, P)$ be the probability space in which the Brownian motion $\{\mathbf{W}_t, 0 \leq t < \infty\}$ is defined, and $\boldsymbol{X}_0$ is the initial value of model (4) and independent of $\mathcal{F}_\infty$. Define the left-continuous filtration $\mathcal{G}_t = \sigma(\boldsymbol{X}_0) \vee \{\mathcal{F}_t^W, 0 \leq t < \infty\}$ as well as the collection of null sets $\mathcal{N} = \{N \in \Omega; \exists G \in \mathcal{G}_\infty \text{ with } N \subseteq G \text{ and } P(G) = 0\}$. Then the augmented filtration is defined as $\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N}),\ 0 \leq t < \infty;\ \mathcal{F}_\infty = \sigma(\bigcup_{t\geq 0}\mathcal{F}_t)$. First note that by stochastic calculus we have $E[\mathcal{B}_i^{k,\ell}|\mathcal{F}_0] = 0$ and for $i \neq j$, $\mathcal{B}_i^{k,\ell}$ and $\mathcal{B}_j^{k,\ell}$ are

independent. Therefore, we only need to verify the conditions of the central limit theorem for the martingale difference array (see, e.g. Hall and Heyde (1980), Corollary 3.1, P.58); namely, we need to check

$$\sum_{i=1}^{n} E\Big(\frac{\sqrt{n}(1-\lambda)}{1-\lambda^n}\lambda^{i-1}\sum_{\ell\le k} c_{k\ell}\mathcal{B}_i^{k,\ell}|\mathcal{F}_{t_i}\Big)^2 \xrightarrow{\text{P}} \frac{\tau(1+e^\tau)}{e^\tau-1}\mathbf{c}^T P_D^T(\mathbf{\Sigma}(f_t)\otimes\mathbf{\Sigma}(f_t))P_D\mathbf{c} \tag{A.15}$$

and

$$\sum_{i=1}^{n} E\Big[\big(\sqrt{n}\frac{1-\lambda}{1-\lambda^n}\lambda^{i-1}\sum_{\ell\le k} c_{k\ell}\mathcal{B}_i^{k,\ell}\big)^4\Big|\mathcal{F}_{t_i}\Big] \xrightarrow{\text{P}} 0. \tag{A.16}$$

Expression(A.15) gives the asymptotic conditional variance of $\sqrt{n}\mathcal{B}$ and (A.16) implies the conditional Lindeberg condition. These two conditions lead to

$$\sqrt{n}\mathcal{B} \xrightarrow{\text{D}} Z_{\mathbf{c}}, \tag{A.17}$$

where the random variable $Z_{\mathbf{c}}$ is defined as in Theorem 1.

We first prove (A.15). From stochastic calculus we know that $E[\mathcal{B}_i^{k,\ell}|\mathcal{F}_{t_i}]=0$ and for $i\neq j$, $\mathcal{B}_i^{k,\ell}$ and $\mathcal{B}_j^{k,\ell}$ are independent. Moreover, by (15) we have

$$\begin{aligned}
E[\mathcal{B}_i^{k_1,\ell_1}\mathcal{B}_i^{k_2,\ell_2}|\mathcal{F}_{t_i}] =& \Delta^{-2}\sum_{j,g} H_{j,g}^{k_1,\ell_1}(f_{t_0})H_{j,g}^{k_2,\ell_2}(f_{t_0})\int_{t_i}^{t_{i+1}} E(W_s^j-W_{t_i}^j)^2 ds\\
=& \frac{1}{2}\sum_{j,g} H_{j,g}^{k_1,\ell_1}(f_{t_0})H_{j,g}^{k_2,\ell_2}(f_{t_0})\\
=& \frac{1}{2}\sum_{j,g} H_{j,g}^{k_1,\ell_1}(f_t)H_{j,g}^{k_2,\ell_2}(f_t)+o_g((n\Delta+\Delta)^q),
\end{aligned}$$

where $H_{j,g}^{k,\ell}(x)=\sigma_{kj}(x)\sigma_{\ell g}(x)+\sigma_{kg}(x)\sigma_{\ell j}(x)$. It follows that

$$\begin{aligned}
\text{var}(\sum_{\ell\le k} c_{\ell k}\mathcal{B}_i^{\ell,k}|\mathcal{F}_{t_i}) &= \mathbf{c}^T P_D(2\mathbf{\Sigma}(f_{t_0})\otimes\mathbf{\Sigma}(f_{t_0}))P_D^T\mathbf{c}\\
&\xrightarrow{\text{P}} \mathbf{c}^T P_D(2\mathbf{\Sigma}(f_t)\otimes\mathbf{\Sigma}(f_t))P_D^T\mathbf{c}.
\end{aligned}$$

Therefore, we get the following result for the conditional variance of the left hand side of (A.15):

$$\begin{aligned}
\sum_{i=1}^{n} E\Big(\frac{\sqrt{n}(1-\lambda)}{1-\lambda^n}\lambda^{i-1}\sum_{\ell\le k} c_{k\ell}\mathcal{B}_i^{k,\ell}|\mathcal{F}_{t_i}\Big)^2 &= \frac{n(1-\lambda)(1+\lambda^n)}{(1+\lambda)(1-\lambda^n)}\text{var}(\sum_{\ell\le k} c_{\ell k}\mathcal{B}_i^{\ell,k}|\mathcal{F}_{t_i})\\
&\xrightarrow{\text{P}} \frac{\tau(1+e^\tau)}{e^\tau-1}\mathbf{c}^T P_D^T(\mathbf{\Sigma}(f_t)\otimes\mathbf{\Sigma}(f_t))P_D\mathbf{c},
\end{aligned}$$

where $\tau = \lim_{n\to\infty} n(1-\lambda)$. This verifies (A.15).

Then we show (A.5). Straightforward calculations yield

$$\begin{aligned}
&E\Big[\big(\sum_{\ell\le k} c_{k\ell}\mathcal{B}_i^{k,\ell}\big)^4\big|\mathcal{F}_{t_i}\Big] = O(1)\sum_{\ell\le k} c_{k\ell}^4 E[(\mathcal{B}_i^{k,\ell})^4|\mathcal{F}_{t_i}] \\
=&O(1)\sum_{\ell\le k} c_{k\ell}^4\Delta^{-4}\sum_{j,g}(H_{j,g}^{k,\ell}(f_{t_0}))^4 E\Big[\big(\int_{t_i}^{t_{i+1}}(W_s^j - W_{t_i}^j)dW_s^g\big)^4\Big|\mathcal{F}_{t_{i-1}}\Big] \\
=&O(1)\sum_{\ell\le k} c_{k\ell}^4\sum_{j,g}(H_{j,g}^{k,\ell}(f_{t_0}))^4.
\end{aligned}$$

This together with Assumption 2 and Hölder's inequality leads to

$$\sum_{i=1}^n E\Big[\big(\sqrt{n}\frac{1-\lambda}{1-\lambda^n}\lambda^{i-1}\sum_{\ell\le k} c_{k\ell}\mathcal{B}_i^{k,\ell}\big)^4\big|\mathcal{F}_{t_i}\Big] = O(n^{-1})\sum_{\ell\le k} c_{k\ell}^4\sum_{j,g}(H_{j,g}^{k,\ell}(f_{t_0}))^4 \xrightarrow{\mathrm{P}} 0,$$

which proves (A.5). (A.17) holds in consequence. Combining (A.14) and (A.17) and applying Slutsky's lemma, we obtain the conclusion in lemma 2. ■

**Lemma 3** *Under Assumptions 1 and 2, as $n\to\infty$ and $n\Delta^q\to 0$, the following result holds for $C_{n,\Delta}$ defined in (A.6)*

$$E\left|C_{n,\Delta} - \mathbf{c}^T vech(\mathbf{\Sigma}(f_t))\right| = O\left((n\Delta)^q\right). \tag{A.18}$$

***Proof***. Note that

$$\begin{aligned}
E|C_{n,\Delta} - \sum_{\ell\le k} c_{k\ell}v_{k\ell,t}| &= \frac{1-\lambda}{1-\lambda^n}E\big|\sum_{i=1}^n\lambda^{i-1}\sum_{\ell\le k}^k c_{k\ell}\left(\bar{v}_{N-i}^{k,\ell} - v_{k\ell,t}\right)\big| \\
&\le \frac{1-\lambda}{1-\lambda^n}\sum_{i=1}^n\lambda^{i-1}\sum_{\ell\le k}^k c_{k\ell}E|\bar{v}_{N-i}^{k,\ell} - v_{k\ell,t}|.
\end{aligned}$$

Thus we only need to consider the asymptotic property of $E|\bar{v}_i^{k,\ell} - v_{k\ell,t}|$. By the Cauchy-Schwarz inequality and Hölder's inequality, we have

$$\begin{aligned}
E\big|\bar{v}_i^{k,\ell} - v_{k\ell,t}\big| \le \Delta^{-1}\sum_{j=1}^m\int_{t_i}^{t_{i+1}} &\big\{E\big|\sigma_{kj}(f_t)\big(\sigma_{\ell j}(f_t) - \sigma_{\ell j}(f_s)\big)\big| \\
&+ E\big|\big(\sigma_{kj}(f_t) - \sigma_{kj}(f_s)\big)\sigma_{\ell j}(f_s)\big|\big\}ds \\
\le \Delta^{-1}\sum_{j=1}^m\int_{t_i}^{t_{i+1}} &\big\{\big[E\sigma_{kj}^2(f_t)E\big(\sigma_{\ell j}(f_t) - \sigma_{\ell j}(f_s)\big)^2\big]^{1/2} \\
&+ \big[E\big(\sigma_{kj}(f_t) - \sigma_{kj}(f_s)\big)^2 E\sigma_{\ell j}^2(f_s)\big]^{1/2}\big\}ds
\end{aligned}$$

Therefore by (15) and Assumption 2,

$$E\big|\bar{v}_i^{k,\ell} - v_{k\ell,t}\big| = O\big((n\Delta+\Delta)^q\big) = O\big((n\Delta)^q\big).$$

This proves (A.18). ■

## A.3 Proof of Proposition 2

**Lemma 4** *(The Tanaka Formula) Let $S_t$ be a continuous semimartingale. For any real number $a$, there exists a non-decreasing continuous process $L_S(\cdot, a)$ called the local time of $S_t$ at $a$, such that*

$$|S_t - a| = |S_0 - a| + \int_0^t sgn(S_s - a)dS_s + L_S(t,a),$$
$$(S_t - a)^+ = (S_0 - a)^+ + \int_0^t 1_{\{S_s > a\}}dS_s + \frac{1}{2}L_S(t,a),$$
$$(S_t - a)^- = (S_0 - a)^- - \int_0^t 1_{\{S_s \le a\}}dS_s + \frac{1}{2}L_S(t,a).$$

*In particular, $|S_t - a|$, $(S_t - a)^+$, and $(S_t - a)^-$ are semimartingales.*

***Proof***. See Revuz and Yor (1999), Theorem 1.2, Chapter 6, p.222. ■

The process $L_S(t,a)$ is called the local time of $S_t$ at point $a$ over time interval $[0,\ t]$. It is measured in units of the quadratic variation process and gives the amount of time that the process spends in the vicinity of $a$.

**Lemma 5** *Since $f_t$ is a stationary real ergodic process, we have*

$$\frac{L_f(T,x)}{\sum b_j^2(x)T} \xrightarrow{a.s.} p(x),$$

*where $p(x)$ is the time-invariant density function of the process $f_t$ at $x$.*

***Proof***. See Bandi and Phillips (2003) and Bosq (1998, Theorem 6.3, P150). ■

**Lemma 6** *Suppose $\Delta \to 0$, $N\Delta \to \infty$, and $\frac{1}{h}\sqrt{\Delta \log \Delta^{-1}} = o(1)$. Under Assumptions 3–5, we have for $\ell = 0, 1, 2, 3$*

$$W_\ell(x) = \frac{1}{\Delta}\int_{t_0}^{t_{N-1}} (f_s - x)^\ell K_h(f_s - x)ds + Nh^{\ell-1}O_{a.s.}\Big(\sqrt{\Delta \log \Delta^{-1}}\Big).$$

***Proof***. First, note that for any nonnegative integer $\ell \leq 4$, we have

$$\begin{aligned}
&\Big|W_\ell(x) - \frac{1}{\Delta}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}(f_s - x)^\ell K\Big(\frac{f_s - x}{h}\Big)ds\Big| \qquad \text{(A.19)}\\
\leq &\frac{1}{h\Delta}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\Big|(f_{t_k} - x)^\ell K\Big(\frac{f_{t_k} - x}{h}\Big) - (f_s - x)^\ell K\Big(\frac{f_s - x}{h}\Big)\Big|ds\\
\leq &I_1 + I_2
\end{aligned}$$

with

$$I_1 = \frac{1}{h\Delta}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\left|K'\Big(\frac{\widehat{f}_{ks} - x}{h}\Big)\right|\left|\frac{f_s - f_{t_k}}{h}\right||f_{t_k} - x|^\ell ds \qquad \text{(A.20)}$$

and

$$I_2 = \frac{1}{h\Delta}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\Big|(\bar{f}_{ks} - x)^{\ell-1}(f_s - f_{t_k})\Big|K\Big(\frac{f_s - x}{h}\Big)ds, \qquad \text{(A.21)}$$

where $\widehat{f}_{ks}$ and $\bar{f}_{ks}$ are both values on the line segment connecting $f_{t_k}$ to $f_s$. Now define

$$\kappa_{N,\Delta} = \max_{i \leq N-1}\sup_{t_{i-1} \leq s \leq t_i}|f_s - f_{t_{i-1}}|.$$

Then, by Levy's modulus of continuity of diffusions (see, e.g. Revuz and Yor (1998, Ch. V, Exercise 1.20)),

$$P\Big(\limsup_{\Delta \to 0}\frac{\kappa_{N,\Delta}}{\sqrt{\Delta \log \Delta^{-1}}} = \alpha\Big) = 1, \qquad \text{(A.22)}$$

where $\alpha$ is a suitable constant. In turn, (A.22) implies that

$$\kappa_{N,\Delta} = O_{a.s.}\Big(\sqrt{\Delta \log \Delta^{-1}}\Big).$$

This together with the assumption that $\frac{1}{h}\sqrt{\Delta \log \Delta^{-1}} = o(1)$ leads to

$$\frac{\kappa_{N,\Delta}}{h} = o_{a.s.}(1) \text{ as } N\Delta \to \infty.$$

In view of (A.20) and (A.21), we have

$$K'\big(\frac{\widehat{f}_{ks} - x}{h}\big) = K'\big(\frac{f_s - x}{h} + o_{a.s.}(1)\big)$$

and

$$\bar{f}_{ks} - x = h\big(\frac{f_s - x}{h} + o_{a.s.}(1)\big),$$

uniformly over $k = 0, \cdots, N-1$. Hence, by Lemma 5 and Revuz and Yor (1999), Exercise 1.15 and Corollary 1.6 of Chapter 6, we obtain that (A.20) can be bounded as

$$\begin{aligned} I_1 \leq & \frac{\kappa_{N,\Delta}}{h} \frac{h^{\ell-1}}{\Delta} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \big|K'\big(\frac{f_s - x}{h} + o_{a.s.}(1)\big)\big| \big|\frac{f_s - x}{h} + o_{a.s.}(1)\big|^{\ell} ds \\ = & N\Delta h^{\ell-1} \frac{\kappa_{N,\Delta}}{h} \int_{-\infty}^{\infty} \big|K'\big(\frac{y - x}{h} + o_{a.s.}(1)\big)\big| \big|\frac{y - x}{h} + o_{a.s.}(1)\big|^{\ell} \frac{L_f(t_{N-1}, y)}{N\Delta \sum b_j^2(y)} dy \\ = & N h^{\ell} \frac{\kappa_{N,\Delta}}{h} \int_{-\infty}^{\infty} \big|K'\big(u + o_{a.s.}(1)\big)\big| |u + o_{a.s.}(1)|^{\ell} (p(uh + x) + o_{a.s.}(1)) du. \end{aligned}$$

This together with (9) yields

$$I_1 \leq N h^{\ell} O_{a.s.}\big(\frac{1}{h}\sqrt{\Delta \log \Delta^{-1}}\big).$$

Similarly, we can show that (A.21) is also bounded by $N h^{\ell} O_{a.s.}\big(\frac{1}{h}\sqrt{\Delta \log \Delta^{-1}}\big)$. This proves the stated results. ■

**Proof of Proposition 2**

Since $x^{2\ell} K(x)$ is a positive function, by Exercise 1.15 and Corollary 1.6 of Chapter 6 of Revuz and Yor (1999), and Lemma 5 above we have for $\ell = 0, 1$,

$$\begin{aligned} & \frac{1}{N\Delta} \int_{t_0}^{t_{N-1}} \big(\frac{f_s - x}{h}\big)^{2\ell} K\big(\frac{f_s - x}{h}\big) ds \\ & = \int \big(\frac{y - x}{h}\big)^{2\ell} K\big(\frac{y - x}{h}\big) \frac{L_f(t_{N-1}, y)}{N\Delta \sum b_j^2(y)} dy \\ & = h \int u^{2\ell} K(u) (p(uh + x) + o_{a.s.}(1)) du \\ & = h\big(p(x)\mu_{2\ell} + o_{a.s.}(1)\big), \end{aligned}$$

where we have used $\mu_4 = \int x^4 K(x) dx < \infty$. This together with Lemma 6 leads to

$$\begin{aligned} \frac{1}{N} W_{2\ell}(x) = & \frac{1}{N\Delta} \int_{t_0}^{t_{N-1}} (f_s - x)^{2\ell} K_h(f_s - x) ds + o_{a.s.}(1) \qquad \text{(A.23)} \\ = & h^{2\ell} (p(x)\mu_{2\ell} + o_{a.s.}(1)). \end{aligned}$$

Let $s(dx) = \exp\Big\{\int_{\alpha}^{x} \frac{2a(y)}{\sum b_j^2(y)} dy\Big\} \frac{2dx}{\sum b_j^2(x)}$ be the speed measure of $f_t$. By the Quotient theorem (Revuz and Yor (1999), Theorem 3.12, Chapter 10, p.427),

$$\begin{aligned} \frac{\frac{1}{N\Delta} \int_{t_0}^{t_{N-1}} \Big(\frac{f_s - x}{h}\Big)^{2\ell+1} K_h(f_s - x) ds}{\frac{1}{N\Delta} \int_{t_0}^{t_{N-1}} K_h(f_s - x) ds} & = \frac{\int \big(\frac{y-x}{h}\big)^{2\ell+1} K_h(y - x) s(dy)}{\int K_h(y - x) s(dy)} + o_{a.s.}(1) \\ & = \frac{\mu_{2\ell+1}}{\mu_0} + o_{a.s.}(1) \end{aligned}$$

as $N\Delta \to \infty$. In turn, this implies that

$$\begin{aligned}\frac{W_{2\ell+1}(x)/h^{2\ell+1}}{W_0(x)} &= \frac{\frac{1}{\Delta}\int_{t_0}^{t_{N-1}} \left(\frac{f_s - x}{h}\right)^{2\ell+1} K_h(f_s - x)ds + NO_{a.s.}\big(\frac{\sqrt{\Delta \log \Delta^{-1}}}{h}\big)}{\frac{1}{\Delta}\int_{t_0}^{t_{N-1}} K_h(f_s - x)ds + NO_{a.s.}\big(\frac{\sqrt{\Delta \log \Delta^{-1}}}{h}\big)} \\ &= \frac{\mu_{2\ell+1}}{\mu_0} + o_{a.s.}(1).\end{aligned} \tag{A.24}$$

Combining (A.23) and (A.24), we obtain

$$W_{2\ell+1}(x) = Nh^{2\ell+1}(p(x)\mu_{2\ell+1} + o_{a.s.}(1)).$$

This completes the proof. ■

## A.4 Proof of Theorem 2

Let $\boldsymbol{M}(f_{t_k}) = E[\boldsymbol{Y}_k \boldsymbol{Y}_k^T | f_{t_k}]$. Then the matrix function $\boldsymbol{M}(y)$ can be expanded around a fixed point $x$ as

$$\boldsymbol{M}(y) = \mathbf{A}_0 + \mathbf{A}_1(y - x) + \mathbf{A}_2(y - x)^2 + \mathbf{A}_3(y - x)^3 + \cdots,$$

where $\mathbf{A}_0,\ \mathbf{A}_1, \cdots$ are all matrices. To prove the asymptotic property of the state-domain estimator, let us decompose it as

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_{S,t}(x) - \boldsymbol{M}(x) &= \sum_{k=0}^{N-1} w_k(x)\left(\boldsymbol{M}(f_{t_k}) - \boldsymbol{M}(x)\right) + \sum_{k=0}^{N-1} w_k(x)\big(\boldsymbol{Y}_k \boldsymbol{Y}_k^T - \boldsymbol{M}(f_{t_k})\big) \\ &\equiv \mathbf{b} + \mathbf{t}.\end{aligned} \tag{A.25}$$

First, we establish the asymptotic behavior of the bias term $\mathbf{b}$. Applying Taylor's expansion and Proposition 2 results in

$$\begin{aligned}\mathbf{b} &= \sum_{k=0}^{N-1} w_k(x)\left(\boldsymbol{M}(f_{t_k}) - \boldsymbol{M}(x)\right) \\ &= \sum_{k=0}^{N-1} w_k(x)\mathbf{A}_1(f_{t_k} - x) + \sum_{k=0}^{N-1} w_k(x)\mathbf{A}_2(f_{t_k} - x)^2 + o_{a.s.}(h^3) \\ &= h^2\mu_2\mathbf{A}_2 + o_{a.s.}(h^2).\end{aligned}$$

Since we have the following decomposition

$$\hat{\boldsymbol{\Sigma}}_{S,t}(x) - \boldsymbol{\Sigma}(x) = \big(\hat{\boldsymbol{\Sigma}}_{S,t}(x) - \boldsymbol{M}(x)\big) + \big(\boldsymbol{M}(x) - \boldsymbol{\Sigma}(x)\big) = [\mathbf{b} + \big(\boldsymbol{M}(\mathbf{x}) - \boldsymbol{\Sigma}(\mathbf{x})\big)] + \mathbf{t},$$

and $\boldsymbol{M}(x) - \boldsymbol{\Sigma}(x) = o_p(\Delta)$, the asymptotic bias of the state-domain estimator is

$$\mathbf{b} + \big(\boldsymbol{M}(x) - \boldsymbol{\Sigma}(x)\big) = \frac{1}{2}h^2\mu_2\boldsymbol{\Sigma}''(x) + o_{a.s.}(h^2) + o_p(\Delta). \tag{A.26}$$

Then, let us consider the variance term $\mathbf{t}$. Since $\mathbf{t}$ is a matrix, we first vectorize it and then consider the asymptotic normality of its linear combination, i.e. we look at the statistic

$$\tilde{\mathbf{t}} = \mathbf{a}^T \mathrm{vech}\Big( \sum_{k=0}^{N-1} w_k(x)\big(\boldsymbol{Y}_k\boldsymbol{Y}_k^T - \boldsymbol{M}(f_{t_k})\big)\Big),$$

where $\mathbf{a}$ is a constant vector. By Proposition 2,

$$\begin{aligned}\tilde{\mathbf{t}} &= \frac{1}{p(x)N}\sum_{k=0}^{N-1} K_h(f_{t_k} - x)a^T\mathrm{vech}\big(\boldsymbol{Y}_k\boldsymbol{Y}_k^T - \boldsymbol{M}(f_{t_k})\big)\{1 + o_{a.s.}(1)\} \\ &\equiv \mathcal{A}_N\{1 + o_{a.s.}(1)\}.\end{aligned} \tag{A.27}$$

Therefore, we only need to show the asymptotic normality of $\mathcal{A}_N$. To this end, first let $\vartheta_{N,k} = K_h(f_{t_k} - x)\mathbf{a}^T\mathrm{vech}\big(\boldsymbol{Y}_k\boldsymbol{Y}_k^T - \boldsymbol{M}(f_{t_k})\big)$. Then $\mathcal{A}_N = \frac{1}{p(x)N}\sum_{k=0}^{N-1}\vartheta_{N,k}$. Straightforward calculations give

$$\begin{aligned}\mathrm{var}(\vartheta_{N,k}) &= E\left(K_h(f_{t_k} - x)\mathbf{a}^T\mathrm{vech}\left(\boldsymbol{Y}_k\boldsymbol{Y}_k^T - \boldsymbol{M}(f_{t_k})\right)\right)^2 \\ &= E\big\{K_h^2(f_{t_k} - x)E\Big[\big(\mathbf{a}^T\mathrm{vech}(\boldsymbol{Y}_k\boldsymbol{Y}_k^T - \boldsymbol{M}(f_{t_k}))\big)^2|f_{t_k}\Big]\big\} \\ &= 2E\left\{K_h^2(f_{t_k} - x)\left(\mathbf{a}^T P_D\boldsymbol{\Sigma}(f_{t_k})\otimes\boldsymbol{\Sigma}(f_{t_k})P_D^T\mathbf{a}\right)\right\} \\ &= 2h^{-1}\nu_0 p(x)\mathbf{a}^T P_D\boldsymbol{\Sigma}(x)\otimes\boldsymbol{\Sigma}(x)P_D^T\mathbf{a}(1 + o(1)),\end{aligned} \tag{A.28}$$

where the last step follows from Taylor's expansion.

Note that $\boldsymbol{Y}_{t_\ell}$ only depends on the sample path of $f_t$ over time interval $[t_\ell,\ t_{\ell+1}]$. Thus by conditioning on $\mathcal{F}_{t_\ell}$, we obtain

$$\mathrm{cov}(\vartheta_{N,1}, \vartheta_{N,\ell+1}) = E\big[\vartheta_{N,1}K_h(f_{t_\ell} - x)E\big(\mathbf{a}^T\mathrm{vech}\big(\boldsymbol{Y}_\ell\boldsymbol{Y}_\ell^T - \boldsymbol{M}(f_{t_\ell})\big)\big|\mathcal{F}_{t_\ell}\big)\big] = 0,\ \ell \geq 1. \tag{A.29}$$

Combining (A.28) and (A.29) entails

$$\mathrm{var}(\mathcal{A}_N) = \frac{2\nu_0}{Nhp(x)}\mathbf{a}^T P_D\boldsymbol{\Sigma}(x)\otimes\boldsymbol{\Sigma}(x)P_D^T\mathbf{a}(1 + o(1)).$$

Since a stationary Markov process satisfying the $G_2$ condition of Rosenblatt (1970) is $\rho$-mixing, we can use "big-block and small-block" arguments similar to those used

by Fan and Yao (2003, Theorem 2.22, p.77) to prove the asymptotic normality of $\mathcal{A}_N$. The lengthy details are omitted here. Thus,

$$\sqrt{Nh}\mathcal{A}_N \xrightarrow{\mathrm{D}} \mathcal{N}(0, 2\nu_0 p(x)^{-1}\mathbf{a}^T P_D \mathbf{\Sigma}(x) \otimes \mathbf{\Sigma}(x) P_D^T \mathbf{a}).$$

This together with (A.26) and (A.27) implies the asymptotic normality of the state-domain estimator, i.e.

$$\sqrt{Nh}\mathbf{a}^T \mathrm{vech}\Big(\hat{\mathbf{\Sigma}}_{S,t}(x) - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\mu_2\mathbf{\Sigma}''(x)\Big) \xrightarrow{\mathrm{D}} \mathcal{N}(0, 2\nu_0 p(x)^{-1}\mathbf{a}^T\Lambda(x)\mathbf{a}),$$

where $\mathbf{a}$ is an arbitrary constant vector. This completes the proof. ■

## A.5 Proof of Theorem 3

We only need to show the asymptotic normality of the linear combination

$$\sqrt{Nh}\ \mathbf{a}^T \mathrm{vech}\left(\widehat{\mathbf{\Sigma}}_{S,t} - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\mu_2\mathbf{\Sigma}''(x)\right) + \sqrt{n}\ \mathbf{c}^T \mathrm{vech}\left(\widehat{\mathbf{\Sigma}}_{T,t} - \mathbf{\Sigma}(x)\right),$$

where $\mathbf{a}^T$ and $\mathbf{c}^T$ are two constant vectors. This is equivalent to showing the joint asymptotic normality of $\sqrt{Nh}\mathbf{a}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{S,t} - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\mu_2\mathbf{\Sigma}''(x)\big)$ and $\sqrt{n}\mathbf{c}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{T,t}\big)$. From the proof of Theorem 2, we have

$$\mathbf{a}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{S,t} - \mathbf{\Sigma}(x) - \frac{1}{2}h^2\mu_2\mathbf{\Sigma}''(x)\big) = \mathbf{a}^T\mathbf{t} + o_p(1) = \tilde{\mathbf{t}} + o_p(1) = \mathcal{A}_N\{1 + o_{a.s.}(1)\} + o_p(1),$$

where $\mathbf{t}$, $\tilde{\mathbf{t}}$ and $\mathcal{A}_N$ are all defined in the proof of Theorem 2. Therefore, we need only to consider about the asymptotic normality of $\sqrt{Nh}\mathcal{A}_N$ and $\sqrt{n}\mathbf{c}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{T,t}\big)$.

We truncate $\mathcal{A}_N$ by defining

$$\mathcal{A}_N^t = \frac{1}{p(x)N}\sum_{k=0}^{N-a_N}\vartheta_{N,k},$$

where $a_N$ is an integer depending only on $N$ and satisfying $a_N/N \to 0$ and $a_N\Delta \to \infty$. We are going to show that:

(i) $\mathcal{A}_N^t$ and $\sqrt{n}\mathbf{c}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{T,t}\big)$ are asymptotically independent;

(ii) $\mathcal{A}_N - \mathcal{A}_N^t$ is asymptotically negligible.

We first prove (i). Since a stationary Markov process satisfying the $G_2$ condition of Rosenblatt (1970) is $\rho$-mixing with exponentially decaying $\rho$-mixing coefficient $\rho_t(\cdot)$, and the strong-mixing coefficient $\alpha(\ell) \leq \rho(\ell)$ for any integer $\ell$, it follows that

$$\left|E\exp\{i\xi(\mathcal{A}_N^t+\mathbf{c}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{T,t}\big))\}-E\exp\{i\xi(\mathcal{A}_N^t)\}E\exp\{\imath\xi\mathbf{c}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{T,t}\big)\}\right| \leq 32\alpha(a_N-n) \to 0,$$

for any $\xi \in \mathbb{R}$. This proves (i).

Now, we prove (ii). From the proof of Theorem 2 we know that

$$\mathrm{var}(\vartheta_{N,k}) = 2h^{-1}\nu_0 p(x)\mathbf{a}^T P_D \mathbf{\Sigma}(x) \otimes \mathbf{\Sigma}(x) P_D^T \mathbf{a}(1+o(1)),$$

and $\mathrm{cov}(\vartheta_{N,1}, \vartheta_{N,\ell+1}) = 0$, $\forall \ell \geq 1$. Therefore,

$$\mathrm{var}(\sqrt{Nh}[\mathcal{A}_N - \mathcal{A}_N^t]) = \frac{2a_N}{p(x)N}\nu_0 \mathbf{a}^T P_D \mathbf{\Sigma}(x) \otimes \mathbf{\Sigma}(x) P_D^T \mathbf{a}(1+o(1)) \to 0.$$

This along with $E[\vartheta_{N,k}] = 0$ gives

$$\sqrt{Nh}[\mathcal{A}_N - \mathcal{A}_N^t] \xrightarrow{\mathrm{P}} 0,$$

which completes the proof of (ii). Combining (i) and (ii) entails that $\sqrt{Nh}\mathcal{A}_N$ and $\sqrt{n}\mathbf{c}^T\mathrm{vech}\big(\widehat{\mathbf{\Sigma}}_{T,t}\big)$ are asymptotically independent. This together with Theorem 1 and the asymptotical normality of $\sqrt{Nh}\mathcal{A}_N$ shown in the proof of Theorem 2 completes the proof of Theorem 3. ■

# FIGURE LEGENDS

**Figure 1**. Illustration of time- and state-domain estimation. (a) The yields of 1-year, 5-year, and 10-year treasury bills from 1962 to 2005. The vertical bar indicates localization in time, and the horizontal bar represents localization in state of the 5-year treasury bill process. (b) Illustration of time-domain smoothing: 1-year yield differences are plotted against 10-year yield differences with the regression line superimposed. (c) Illustration of the state-domain smoothing: 1-year yield differences are plotted against 10-year yield differences for those periods with the corresponding 5-year yields restricted to the interval $6.37\% \pm .2\%$, indicated by the horizontal bar in (a).

**Figure 2**. Functions $A(\tau)$ (solid curve) and $B(\tau)$ (dashed curve) for the parameters given in the simulation.

**Figure 3**. (a) The averages of the entropy losses over 500 simulations for the time-domain estimation (dotted curve), state-domain estimation (dashed curve), and aggregated method (solid curve). (b) The standard deviations of the entropy losses over 500 simulations for time-domain estimation (dotted curve), state-domain estimation (dashed curve), and the aggregated method (solid curve). (c) and (d): The same as in (a) and (b) except using the quadratic loss.

**Figure 4**. (a) Box plots of the entropy losses over 500 simulations for the time-domain estimator (left), the aggregated method (middle), and the state-domain estimator (right). (b) and (c): The same as in (a) except that the quadratic loss and PE are used, respectively. (d) The ratios of the averages of the quadratic losses over 150 out-sample forecastings using the time-domain and state-domain estimators to those based on the aggregated estimator ($x$-axis) are plotted against the ratios of the PEs based on the time-domain and state-domain estimators to those based on the aggregated estimator ($y$-axis).

**Figure 5**. Correlation of the time-domain estimator and state-domain estimator for the volatility of an equally weighted portfolio. The dashed curves are for the 95% confidence intervals. The straight lines are acceptance regions for testing the null hypothesis that the correlation is zero at significance level 5%.

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# FOOTNOTE

**Footnote 1**. By "stationarity" we do not mean that the process is strongly stationary, but has some structural invariability over time. For example, the conditional moment functions do not vary over time.

**Footnote 2**. Ledoit and Wolf (2003) introduce a shrinkage estimator by combining the sample covariance estimator with that derived from the CAPM. Their procedure intends to improve estimated covariance matrix by pulling the sample covariance towards the estimate based on the CAPM. Their basic assumption is that the return vectors are i.i.d. across time. This usually holds approximately when the data are localized in time. In this sense, their estimator can be regarded as a time-domain estimator.

**Footnote 3**. We prove in Section 4 that $\widehat{\mathbf{\Sigma}}_{S,t}$ and $\widehat{\mathbf{\Sigma}}_{T,t}$ are asymptotically independent, and thus they are close to be independent in finite sample. In the following, by "nearly independent" and "almost uncorrelated", we mean the same.

**Footnote 4**. In practice, one can take the yields process with median term of maturity as the driving factor, as this bond is highly correlated to both short-term and long-term bonds.

**Footnote 5**. The kernel function is a probability density, and the bandwidth is its associated scale parameter. Both of them are used to localize the linear regression around the given point $x_0$. The commonly used kernel functions are the Gaussian density and the Epanechnikov kernel $K(x) = 0.75(1 - x^2)_+$.

**Footnote 6**. The stationarity condition of $f_t$ in Assumption 3 can be weakened to Harris recurrence. See Bandi and Moloche (2004) for asymptotic normality of local constant estimator under recurrence assumption.

**Footnote 7**. The optimal choice of weight is proportional to the effective number of data points used for the state-domain and time-domain smoothing. It always outperforms the choice with $\omega_t = 1$ (state-domain estimator) or $\omega_t = 0$ (time-domain estimator).

**Footnote 8**. The choice comes from the recommendation of the RiskMetrics of J.P. Morgan. The parameter $\lambda$ can also be chosen automatically by data by using the prediction error as in Fan, Jiang, Zhang and Zhou (2003). Since we compare the relative

performance between the time-domain estimator and the aggregated estimator, we opt for this simple choice. The results do not expect to change much when a data-driven technique is used.

**Footnote 9**. Here we add normal noise to make the model more realistic. Our method performs even better without noise. Since the noise vectors are i.i.d. across time and the standard deviations are small, adding them to the original time series does not change the whole structure. Hence, our theory can carry through under contamination.

**Footnote 10**. With $\lambda = 0.94$, the last data point used in the time domain has an extra weight $0.94^{104} \approx 0.0016$, which is very small. Hence, we essentially include all the effective data points.

**Footnote 11**. Europe used several common currencies prior to the introduction of the Euro. The European Currency Unit (ECU) was used from January 1, 1979 to January 1, 1999, when the Euro replaced the European Currency Unit at par.

# TABLE

**Table 1**

APEs of Bond Yields, Exchange Rates and Simulations

| | Time | State | Aggregated |
|---|---|---|---|
| Bonds | | | |
| $k = 0$ | $3.837 \times 10^{-3}$ | $3.767 \times 10^{-3}$ | $3.756 \times 10^{-3}$ |
| $k = 1$ | $1.643 \times 10^{-3}$ | $1.557 \times 10^{-3}$ | $1.555 \times 10^{-3}$ |
| $k = 2$ | $1.013 \times 10^{-3}$ | $1.011 \times 10^{-3}$ | $9.933 \times 10^{-4}$ |
| Currencies | | | |
| $k = 0$ | $4.795 \times 10^{-3}$ | $4.913 \times 10^{-3}$ | $4.755 \times 10^{-3}$ |
| $k = 1$ | $1.681 \times 10^{-3}$ | $1.855 \times 10^{-3}$ | $1.652 \times 10^{-3}$ |
| $k = 2$ | $8.979 \times 10^{-4}$ | $1.184 \times 10^{-3}$ | $8.937 \times 10^{-4}$ |
| Simulations ($k = 0$) | $1.850 \times 10^{-2}$ | $1.846 \times 10^{-2}$ | $1.825 \times 10^{-2}$ |

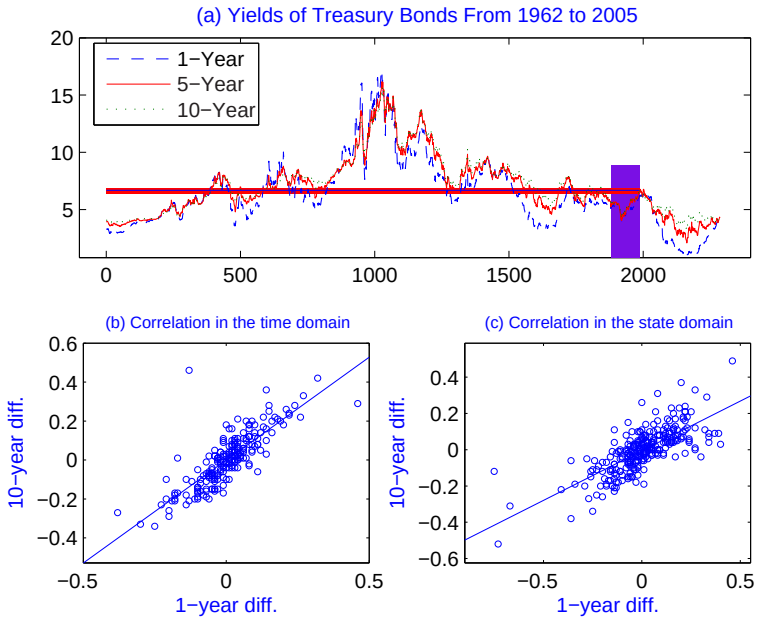



Figure 1: Illustration of time- and state-domain estimation. (a) The yields of 1-year, 5-year and 10-year treasury bills from 1962 to 2005. The vertical bar indicates localization in time, and the horizontal bar represents localization in the state of the 5-year treasury bill process. (b) Illustration of time-domain smoothing: 1-year yield differences are plotted against 10-year yield differences with the regression line superimposed. (c) Illustration of the state-domain smoothing: 1-year yield differences are plotted against 10-year yield differences for those periods with the corresponding 5-year yields restricted to the interval $6.37\% \pm .2\%$, indicated by the horizontal bar in (a).

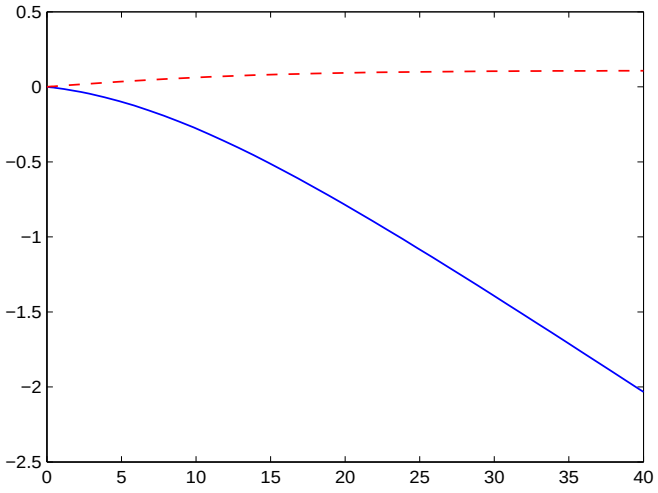


Figure 2: Functions $A(\tau)$ (solid curve) and $B(\tau)$ (dashed curve) for the parameters given in the simulation.

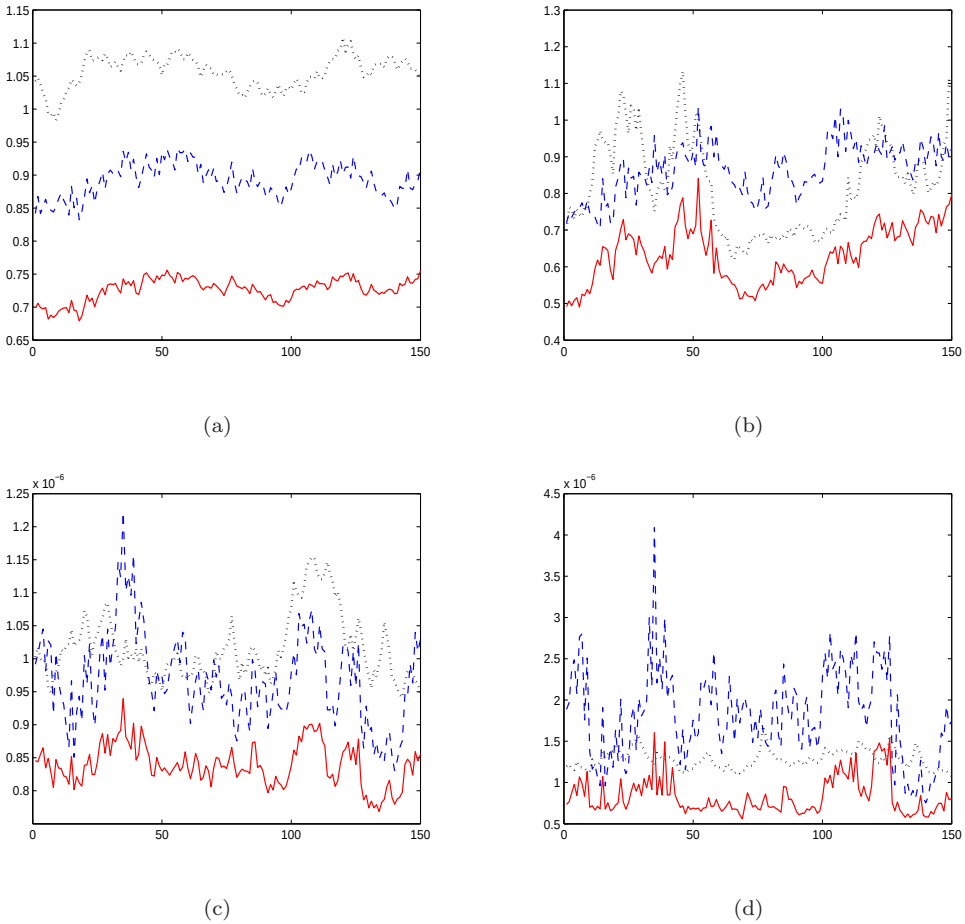
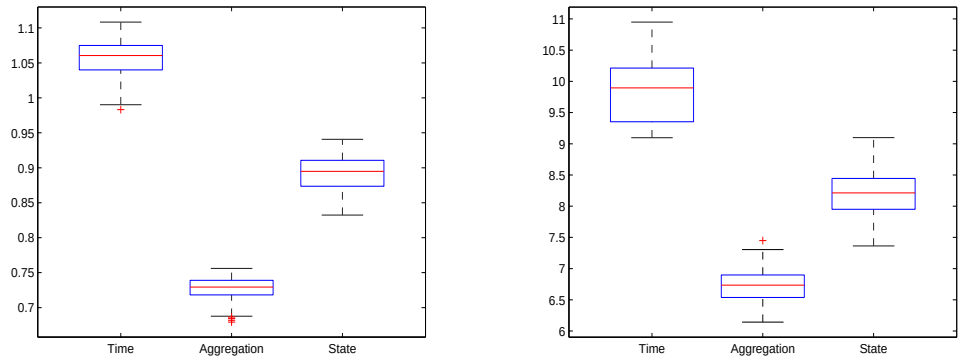


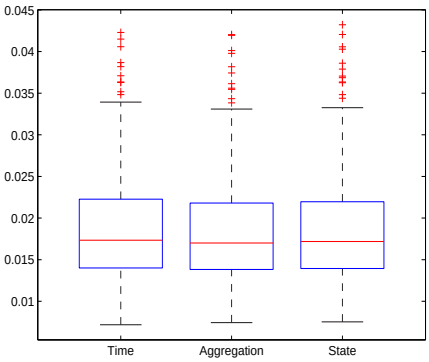

Figure 3: (a) The averages of the entropy losses over 500 simulations for the time-domain estimation (dotted curve), state-domain estimation (dashed curve) and aggregated method (solid curve). (b) The standard deviations of the entropy losses over 500 simulations for the time-domain estimation (dotted curve), state-domain estimation (dashed curve) and aggregated method (solid curve). (c) and (d): The same as in (a) and (b) except using the quadratic loss.



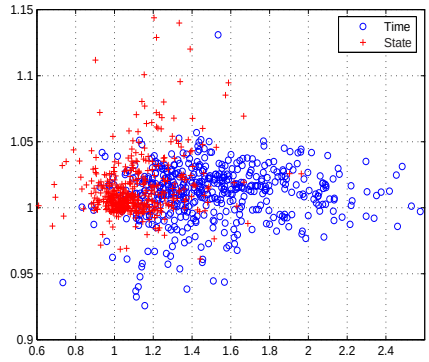

(a)




(b)




(c)




(d)

Figure 4: (a) Box plots of the entropy losses over 500 simulations for the time-domain estimator (left), the aggregated method (middle), and the state-domain estimator (right). (b) and (c): The same as in (a) except that the quadratic loss and PE are used, respectively. (d) The ratios of the averages of the quadratic losses over 150 out-sample forecastings using the time-domain and state-domain estimators to those based on the aggregated estimator ($x$-axis) are plotted against the ratios of the PEs based on the time-domain and state-domain estimators to those based on the aggregated estimator ($y$-axis).

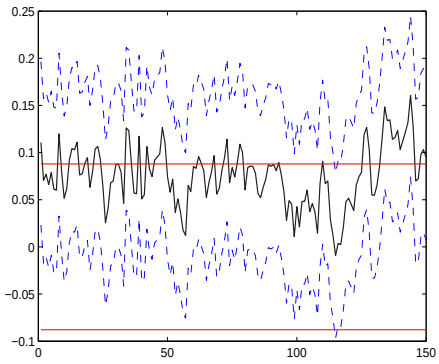



Figure 5: Correlation of the time-domain estimator and state-domain estimator for the volatility of an equally weighted portfolio. The dashed curves are for the 95% confidence intervals. The straight lines are acceptance regions for testing the null hypothesis that the correlation is zero at significance level 5%.