ORF/FIN 504: Financial Econometrics

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Problem Set #2 Spring 2025

Due Friday, February 21, 2025.

1. After fitting an AR(3) model to the monthly log-return of CRSP data from Jan. 1926 to Dec. 1997, it was obtained that

$$\hat{b}_0 = 0.0103, \hat{b}_1 = 0.104, \hat{b}_2 = -0.010, \hat{b}_3 = -0.120$$

with the estimated covariance matrix as follows:

$$\mathbf{S} = 1000^{-2} \begin{pmatrix} 2^2 & 34 & 0 & 0 \\ 34 & 34^2 & 0 & 0 \\ 0 & 0 & 34^2 & 0 \\ 0 & 0 & 0 & 34^2 \end{pmatrix}$$

   (a) What is the standard error of $\hat{b}_1$?

   (b) Test $H_0 : b_1 = 0$ at significant level 1%.

   (c) The annual return is estimated as $\hat{r} = (1+\hat{\mu})^{12} - 1$, where $\hat{\mu} = \hat{b}_0/(1-\hat{b}_1-\hat{b}_2-\hat{b}_3)$. Construct 95% confidence interval of the annual return (computing directly from the information given above)

   (d) Obtain directly the standard error of the annual return $\hat{r}$ if $SE(\hat{\mu}) = 0.002186$ was already computed.

2. Suppose that the daily simple-returns of a stock follow the ARMA model

$$X_t = 0.1X_{t-1} + \varepsilon_t - 0.2\varepsilon_{t-1}, \quad \sigma = 0.1.$$

   (a) Compute $\text{cov}(X_t, \varepsilon_t)$ and $\text{var}(X_t)$.

   (b) Express the ARMA model as an MA(∞) model.

   (c) Given $\varepsilon_{99} = 0.01$, $\varepsilon_{100} = -0.02$ and $X_{99} = 0.2$, compute the one-step and two-step prediction at time $t = 100$.

   (d) Give the associated prediction errors in (c).

3. Show the following facts:

   (a) **(Not required to hand in)** The autocorrelation function of AR(p) process admits the following form:

$$\gamma(k) = \alpha_1 z_1^{-k} + \cdots + \alpha_p z_p^{-k},$$

where $z_1, \cdots, z_p$ are the distinct roots of the characteristic function and $\{\alpha_j\}_{j=1}^k$ are constants. **Hint**: The characteristic polynomial of the AR process is $b(z) = (1 - z/z_1)\cdots(1 - z/z_p)$ and use the Yule-Walker equation $b(B)\gamma(k) = 0$.

(b) For the AR(p) model

$$X_t = b_0 + b_1 X_{t-1} + \cdots + b_p X_{t-p} + \varepsilon_t,$$

show that $\pi(k) = 0$ for $k > p$.

(c) The above $AR(p)$ model is stationary when $\sum_{j=1}^p |b_j| < 1$.

4. Let $X_t = b_0 + b_1 X_{t-1} + \cdots + b_p X_{t-p} + \varepsilon_t + a_1 \varepsilon_{t-1} + \cdots + a_q \varepsilon_{t-q}$ be a stationary ARMA(p, q) process.

   (a) What is the expected value of the process EX_t?

   (b) What is the best two-step prediction? You may assume that realized noises $\{\varepsilon_t\}_{t=1}^T$ are known at time T.

   (c) What is the prediction error of the best two-step ahead forecasting?

5. Suppose the prices of a stock follow the AR(1) model: $X_t = \mu + \rho X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\} \sim_{i.i.d.} N(0, \sigma^2)$ and σ is unknown.

   (a) Derive the conditional maximum likelihood estimator of $\tilde{\mu}$ and $\tilde{\rho}$ for μ and ρ. (Here, the conditional likelihood refers to the density of $(X_2, \cdots, X_T)$ given X_1).

   (b) Use simulation (1000 times with the initial value $X_0 = 0$) to calculate the 1%-tile and 5%-tile of the null distributions of the Dickey-Fuller tests (without drift and with drift) for $T = 50$ and 100 and 200. For concreteness, set $\mu = 0$ and $\sigma = 1$ in your simulation experiment.

6. The monthly closing prices from January 1990 to January, 2019 of the Merck company are given at class website (`http://orfe.princeton.edu/~jqfan/fan/classes/504.html`). Analyze the monthly data with necessary supporting tables and figures. Answer the following questions.

   (a) Do the log-prices follow a random walk?

   (b) Are the log-returns predictable? Use the Ljung-Box test with lags 5 and 10 at significance level 1%.

   (c) Fit an ARMA(p,q) ($p + q \leq 2$) model to the data with order chosen by AIC.

   (d) Check if the residual series is white noise by ACF plot and Ljung-Box test $Q(5)$.